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THE ELEMENTS

OF

HYDROSTATICS.

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THE

ELEMENTS

OF

HYDROSTATICS:

WITH THEIR APPLICATION

TO THE

Solution of Problems.

DESIGNED FOR THE USE OF STUDENTS
IN THE UNIVERSITY.

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PREFACE.

The following pages contain the substance of an Elementary Course of Lectures given in St. John's College, and originally drawn up for the use of a portion of the students of that society. The want of some treatise, to serve as a Text Book, having been found to discourage the pupil and frequently to cause this branch of philosophy to be neglected, occasioned a recommendation from some friends engaged in the tuition, to commit this to the press. It is, therefore, now submitted, with diffidence, to the judgement of the public; and should it be found to facilitate the progress of the students, for whose use it is principally intended, the purpose of its publication will be answered.

In explaining the principles and elements of any branch of philosophy, originality cannot be expected. To arrange and elucidate the discoveries of others, and in some cases to supply deficiences, will generally be the aim of those who are engaged in instruction. But to ensure success, it is peculiarly important that the principles should be fixed in the memory by their application to Examples and Problems; and with the hope of effecting this a considerable number have been here introduced.

The term *Hydrostatics,* in its proper acceptation, signifies that division of the science which treats of the equilibrium of non-elastic fluids; and *Hydrodynamics* that which relates to their forces and motion: in this respect the terms correspond with those of *Statics* and *Dynamics* as applied to solid bodies. But it is not unusual to include the whole doctrine under the general term Hydrodynamics, and to denote the divisions relative to their equilibrium and motion by the terms Hydrostatics and Hydraulics. That part of the science which treats of the mechanical properties of air and the different elastic fluids, is called *Pneumatics.* These divisions are here comprehended under the general name *Hydrostatics;* and it is adopted partly because it is the name in common use, and partly because the scientific distinctions are not closely observed in the arrangement, which has been regulated by considerations of convenience, and what experience has taught to be the easiest mode of instruction.

The first Section contains general principles and such propositions as serve to explain the nature of specific gravity. The second treats of the pressure of fluids; and the third of floating bodies and the modes of thence deducing the specific gravities of solids and fluids. The fourth is occupied with the motion of fluids issuing through the orifices of vessels. To this section it was intended to annex the general equations which have been deduced: but the motion of fluids is a subject involved in such difficulties, and so complicated are the formulæ, when the necessary considerations are introduced, that in a treatise intended for beginners, it has been deemed expedient to omit them altogether.—In the fifth Section will be found the common principles of resistances; followed in the sixth by the motion of wheels, &c. impelled by water, the screw of Archimedes, and the motion of water in canals. The seventh treats of the nature and properties of elastic fluids; the eighth of the thermometer; and the ninth of the expansion and contraction of fluids and solids, and the corrections thereby afforded to the methods of determining specific gravities. In the tenth are explained the effects of the air's pressure in the case of the barometer, pumps, &c. In the eleventh, the motion of elastic fluids, particularly the air; and in the twelfth, the theory of capillary tubes.

It has not been deemed necessary to enumerate all the authors from whom assistance has been derived, or whose observations have been adopted or altered to suit the plan of the present work. Should it fall into the hands of proficients in the science, they will easily discover them: and to the mere beginner it can be of no service to quote a list of names. But so much has been taken from the "Traité d'Hydrodynamique" of Bossut, and the "Traité de Physique" of Biot, that the obligation cannot be passed over without particular acknowledgment.

To the Syndics of the University Press, this opportunity is taken of gratefully acknowledging the liberality with which, from the funds at their disposal, they have defrayed a considerable portion of the expence of printing.

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ELEMENTS

OF

HYDROSTATICS.

SECT. I.

1. DEF. A *Fluid* is a collection of very minute material particles, which cohere so slightly together, that they yield to the smallest force, and by yielding are put in motion amongst each other in every direction.

2. The extreme facility with which the particles move amongst each other in every direction, has generally been supposed to arise from their spherical form, and from the repulsive power exerted by each on all sides. But how this facility of moving is affected under different circumstances, and under different degrees of compression, cannot be ascertained.

3. That the facility with which the component particles move amongst each other is different in different fluids, is evident; or, that different fluids have different degrees of fluidity: and hence they have sometimes been divided into *perfect* and *imperfect:* the former of which are those, between whose component particles there is no sensible cohesion; and the latter, such as oil, &c. where the cohesion is sensible. This division, however, is arbitrary; there being no body which possesses the character of perfect fluidity, that is, whose parts are wholly free from friction, cohesion, and tenacity. Mercury and water are reckoned amongst those which approach most nearly to *perfect:* and boiling water is nearer to a state of perfect fluidity, than water in any other state;

its *viscidity* increasing as the temperature decreases. Water and mercury also, when mixed with heterogeneous substances, become less fluid.

4. Fluids have also been divided into *compressible* and *incompressible*; the former of which are diminished in magnitude by pressure, as air, steam, &c.: and the latter are not affected in magnitude by any increase or diminution of pressure. They may assume an infinite variety of figure; but under all forms the magnitude remains the same.

From the Florentine experiment, water had been supposed incompressible, as well as wine and mercury: they are, however, found to be sensible of heat and cold: and experiments have clearly shewn that water is more compressible in winter than summer; though the contrary is the case in spirits of wine, and oil of olives.

5. All bodies have pores more or less minute. These are easily distinguishable in solids; and though in fluids they cannot be made perceptible to the eye, the fact may be proved by experiments. But what ratio exists between the quantity of empty space and the quantity of matter in any body, cannot be ascertained. *Perfectly dense* matter being unknown, we can only compare with each other the densities of different bodies.

It seems by no means unreasonable to infer, that no fluids are *absolutely* incompressible: for all bodies being porous, their parts may be brought nearer to each other; and therefore a fluid, which is a collection of material particles, be compressible. Hence, the common division of fluids into compressible and incompressible, is, strictly speaking, inaccurate. But it is sometimes necessary to examine extreme cases, in order to determine the relative effects of the qualities of bodies, and to assign to each its proper functions. And as the compression of several fluids is very small, respect being had to their mass, in practical cases it may generally be neglected; and they may still be considered as incompressible, in common investigations.

6. The parts of a fluid have weight, and by their weight press upon each other, and upon other bodies.

For if an empty vessel in one scale be balanced by a weight in the opposite scale, upon pouring some fluid into the vessel it preponderates; the fluid therefore has weight. The parts also press upon each other: for if a vessel, containing a portion of fluid, be balanced; upon pouring more fluid into the vessel, it preponderates: the parts therefore of the fluid press upon each other. They also press upon other bodies: for in either of the preceding experiments it is evident that the fluid presses upon the base of the vessel which contains it.

7. Fluids gravitate in *proprio loco;* that is, they possess the same weight in fluids similar to themselves, that they do in air.

For let a vessel when exhausted be weighed in air, and again when filled with water; the difference of these weights will be the weight of so much water in air. Again, let the vessel when exhausted be weighed in water, and also when filled with water and freely communicating with the surrounding fluid: the difference of these weights is the weight of the same quantity of water in water. Now, it is found that the last difference is equal to the former: water therefore possesses the same weight in water as in air. And the same experiment may be applied to other fluids.

8. Def. The *Specific Gravity* of a body is the relation of its weight, compared with the weight of some other body of the same magnitude.

Thus, brass is said to have 8 times the specific gravity of water, because one cubic inch of brass contains 8 times the quantity of matter, or is 8 times heavier than a cubic inch of water. And the specific gravities of gold and silver are as 19 : 11, because, if equal magnitudes of the two metals be taken, their weights are as 19 : 11.

Also, if a vessel when empty weighs w, when filled with water w', and when filled with any other fluid w''; the weight of the water is $w' - w$, and of the fluid $w'' - w$, and the magnitudes being the same, the ratio of the specific gravities is $w' - w : w'' - w$; and if the specific gravity of water be 1, that of the fluid will be $\dfrac{w'' - w}{w' - w}$.

9. The density and specific gravity of a body are severally as its quantity of matter, when its magnitude is given.

In general, (by mechanics,) $D \times M \propto Q$,

$$\text{therefore, when } M \text{ is given, } D \propto Q.$$

Also from the definition, M being given, $S \propto W$; and the accelerating force of gravity being constant at the same place,

$$W \propto Q; \therefore S \propto Q.$$

If the accelerating force of gravity be not given; $W \propto$ the moving force $\propto Q \times F$, F being the accelerating force,

$$\therefore S \propto Q \times F.$$

10. Cor 1. The specific gravity varies as the density: since each varies as the quantity of matter, when the magnitude is given.

If the accelerating force of gravity is not given, $S \propto D \times F$.

11. Cor. 2. The weight varies as the magnitude and specific gravity jointly.

$$\text{For, } Q \propto M \times D; \text{ but } W \propto Q, \text{ and } S \propto D \quad (10);$$
$$\therefore W \propto M \times S.$$

Ex. Let the diameters of two globes be as 2 : 3, and their specific gravities as 1 : 3. To compare their weights.

The magnitudes of globes being as the cubes of their diameters,

$$M : m :: 8 : 27,$$
$$\text{and } S : s :: 1 : 3;$$
$$\therefore W : w :: 8 : 81 :: 1 : 10.125.$$

12. Cor. 3. Hence, $S \propto \frac{W}{M}$.

The weight of a body, strictly speaking, cannot be divided by its magnitude, the two quantities not being similar, and therefore not capable of comparison. But if the weights and magnitudes of different bodies be referred to similar units, and expressed numerically, viz. the former to the weight, and the latter to the magnitude of a cubic inch of water; any number in the one being divided by a corresponding number in the other, the quotient will represent the specific gravity of the body to which the number belongs.

Ex. 1. Let the diameters of two globes be as 2 : 3, and their weights as 1 : 5. To compare their specific gravities.

$$S : s :: \frac{W}{M} : \frac{w}{m} :: \frac{1}{8} : \frac{5}{27} :: 27 : 40.$$

Ex. 2. If their weights are as 10 : 1, and their specific gravities as 5 : 4. To compare their diameters.

$$\text{Here } M \propto \frac{W}{S}; \therefore M : m :: \frac{10}{5} : \frac{1}{4} :: 8 : 1,$$

$$\text{and } D : d :: 2 : 1.$$

13. Cor. 4. The specific gravities of bodies are inversely as their magnitudes, when the weights are equal.

For $S \propto \frac{W}{M} \propto \frac{1}{M}$, when W is given.

Ex. 1. If the specific gravities of two bodies, A and B, are as 14 : 1, and a cylinder of A, of a certain weight, is 30 inches high; a cylinder of B, of equal base, and having the same weight must be 14 times as large, that is, 14 times as high; or its height must be 35 feet.

Ex. 2. A globe and a cylinder that would circumscribe it, are each formed of a different substance; what is the ratio of their specific gravities, the weight of the two being the same.

Since the magnitudes are as 2 : 3, the specific gravities, which are inversely as the magnitudes, will be as 3 : 2.

14. Cor. 5. The weights of bodies are proportional to their magnitudes, when their specific gravity is the same.

For (11), $W \propto M \times S \propto M$, when S is given.

Ex. A mass of gold immersed in a cylinder of water causes it to rise (a) inches; a mass of silver of the same weight causes it to rise (b) inches; and a mixture of gold and silver of the same weight (c) inches. Find the proportion of gold and silver in the compound.

Let x and y be the magnitudes of gold and silver in the compound; then

$c : a :: x+y$: the magnitude of the gold mass $= \frac{a}{c} \cdot (x+y)$.

In the same way, the magnitude of the silver mass $= \frac{b}{c} \cdot (x+y)$.

Let the weight of each of these bodies be represented by W, then $\frac{a}{c} \cdot (x+y) : x :: W$: the weight of gold in the compound

$$= \frac{cWx}{a \cdot (x+y)},$$

and in the same manner, the weight of silver in the compound

$$= \frac{cWy}{b \cdot (x+y)};$$

$$\therefore \frac{cWx}{a \cdot (x+y)} + \frac{cWy}{b \cdot (x+y)} = W,$$

$$\text{whence, } bcx + acy = ab \cdot (x+y),$$

$$\text{and } (bc - ba) \cdot x = (ab - ac) \cdot y;$$

$$\therefore x : y :: a \cdot (b-c) : b \cdot (c-a).$$

15. Having given the magnitudes and specific gravities of two bodies; to find the specific gravity of the compound formed by their mixture.

Let M and M' represent the magnitudes, S and S' the specific gravities respectively of the simples, and σ the specific gravity of the compound. Since (11) $W \propto M \times S$, suppose it $= c \,.\, M \times S$, (c being some constant quantity, *Algebra* 200); the weight of $M = c \,.\, M \times S$; and the weight of $M' = c \,.\, M' \times S'$; and of the compound $= c \,.\, (M + M') \,.\, \sigma$; whence, the sum of the weights of the simples : the weight of the compound

$$:: c \,.\, M \times S + c \,.\, M' \times S' : c \,.\, (M + M') \,.\, \sigma$$
$$:: M \times S + M' \times S' : (M + M') \,.\, \sigma,$$

but the first term in the proportion being equal to the second,

$$MS + M'S' = (M + M') \,.\, \sigma;$$
$$\therefore \sigma = \frac{MS + M'S'}{M + M'}.$$

16. Cor. 1. If two bodies be mixed together, the magnitude of the heavier is to that of the lighter, as the difference between the specific gravities of the compound and the lighter, is to the difference between the specific gravities of the heavier and the compound.

For since (15), $MS + M'S' = (M + M') \,.\, \sigma$,

$$M \,.\, (S - \sigma) = M' \,.\, (\sigma - S');$$
$$\text{and } \therefore M : M' :: \sigma - S' : S - \sigma.$$

Ex. In an heterogeneous body consisting of two parts, let the specific gravities of those parts and of the whole, be as 8, 3, 7; compare the magnitudes of the parts,

$$M : M' :: 7 - 3 : 8 - 7 :: 4 : 1.$$

17. Cor. 2. Also the magnitude of the compound is to the magnitude of *either* body, as the difference between

the specific gravities of the two bodies, is to the difference between the specific gravities of the compound and of the other.

$$\text{For since (16), } M : M' :: \sigma - S' : S - \sigma;$$

$$\therefore (M + M' =)\, P : M :: S - S' : \sigma - S',$$

$$\text{and } P : M' :: S - S' : S - \sigma.$$

18. If the weights and specific gravities of two bodies be given; to determine the specific gravity of the compound formed by their mixture.

Let W and W' represent the weights, and S and S' the specific gravities of the simples, and σ that of the compound; then as in Art. 15, it may be shewn that

$$\frac{W}{S} + \frac{W'}{S'} = \frac{W + W'}{\sigma};$$

$$\text{and } \therefore \sigma = \frac{(W + W') . SS'}{WS' + W'S}.$$

19. Cor. 1. If equal weights be mixed, $\sigma = \dfrac{2\,SS'}{S + S'}$; but if equal magnitudes (15), $\sigma = \dfrac{S + S'}{2}$, being in the first case an harmonic, in the second an arithmetic mean between S and S'.

20. Cor. 2. To determine the weights of each of the bodies in the mixture, supposed to be known.

From (18), it appears that $WS' . (S - \sigma) = W'S . (\sigma - S')$;

$$\therefore W : W' :: S . (\sigma - S') : S' . (S - \sigma),$$

$$\text{and } W : W + W' :: S . (\sigma - S') : \sigma . (S - S'),$$

$$\text{also } W' : W + W' :: S' . (S - \sigma) : \sigma . (S - S').$$

21. It has here been taken for granted, that the magnitude of the compound is exactly equal to the sum of the

magnitudes of the two ingredients. This, however, is not universally the case; an increase or diminution of magnitude often attending the combination of two different ingredients: thus a cubical inch of alcohol, combined with a cubical inch of water, forms a compound which will measure less than two cubical inches: and a cubical inch of tin in a fluid state, mixed with one of lead, forms a compound exceeding two cubical inches in magnitude. But this consideration belongs rather to chemistry.

22. That the specific gravities of bodies may be more readily compared, it is expedient to assume the specific gravity of some substance as a standard. That of any fluid or solid might be made use of, for this purpose: but in order that the comparison should be made under the same circumstances, it is necessary that the substance should be always the same in respect of its nature and density. Distilled water has been generally taken, at a temperature of 39° Fahrenheit, (a certain temperature being necessary, since heat causes a variation in magnitude); or where this cannot readily be procured, rain water, which has sensibly the same degree of purity, and of which we have every reason to believe the density, when at the same temperature, invariable. It is also particularly convenient for the purpose, as a cubic foot of it weighs 1000 ounces avoirdupois.

Let the specific gravity therefore of water be considered as a standard, and represented by 1000; to determine, on the same scale, the specific gravity S of any other body, whose weight in avoirdupois ounces is W, and magnitude in cubic feet M.

$$\text{Since (11), } W \propto MS, \quad 1000 : W :: 1 \times 1000 : MS;$$
$$\therefore\ W = MS,$$

from which equation the specific gravity may be found.

If, as is the case in tables of specific gravities, that of water be assumed = 1, those of other bodies will be determined by moving the decimal point three places farther to the left.

23. Cor. Hence $M = \frac{W}{S}$; and therefore however irregular the shape of a body may be, if the weight and specific gravity be known, the magnitude in cubic feet may be determined by dividing the weight in avoirdupois ounces by the specific gravity.

Prob. The values of an ounce of platina, gold and silver being p, g, s, and their specific gravities a, b, c; compare the value of a coin made of platina and silver, and which is equal to a guinea in weight and magnitude, with the value of a guinea.

Let x and y be the weights of platina and silver respectively in the coin, and z the weight of a guinea;

$$\therefore x + y = z.$$

Also (23), the magnitudes of platina and silver respectively in the coin will be $\frac{x}{a}$ and $\frac{y}{c}$, and that of a guinea $\frac{z}{b}$;

$$\left.\begin{aligned} \therefore \frac{x}{a} + \frac{y}{c} &= \frac{z}{b} \\ \text{but } \frac{x}{c} + \frac{y}{c} &= \frac{z}{c} \end{aligned}\right\}$$

$$\text{whence } (ab - bc) \cdot x = (ab - ac) \cdot z,$$

$$\text{and } x = \frac{a \cdot (b - c)}{b \cdot (a - c)} \cdot z.$$

In the same manner $y = \frac{c \cdot (a - b)}{b \cdot (a - c)} \cdot z$.

Hence, the value of the platina in the coin $= \frac{a \cdot (b - c)}{b \cdot (a - c)} \cdot pz$,

and the value of the silver $= \frac{c \cdot (a - b)}{b \cdot (a - c)} \cdot sz$;

therefore the value of the coin $= \dfrac{a.(b-c).pz + c.(a-b).sz}{b.(a-c)}$,

and the value of a guinea $= gz$;

therefore the values of the coin and guinea are

$$:: a.(b-c).pz + c.(a-b).sz : b.(a-c).gz$$

$$:: ap.(b-c) + cs.(a-b) : bg.(a-c).$$

24. If with a constant magnitude of water small equal quantities of a lighter fluid be mixed; to determine the variations in the specific gravity of the compound.

Let W represent the weight of the constant quantity of water, and w the weight of the small addition of the lighter fluid, then W will represent the magnitude of that quantity of water (22), let m represent that of the fluid.

The weight of the mixture is $W+w$, and its magnitude $W+m$; therefore (22), its specific gravity

$$= \frac{W+w}{W+m} = 1 + \frac{w-m}{W+m}.$$

When the second portion of fluid is added, the weight is $W+2w$; and supposing the fluid to retain its density, its magnitude is $W+2m$;

therefore its specific gravity $= \dfrac{W+2w}{W+2m} = 1 + \dfrac{2.(w-m)}{W+2m}$,

and after (n) additions, the specific gravity will be

$$= 1 + \frac{n.(w-m)}{W+nm}.$$

25. Cor. 1. If the denominator were constant, the successive specific gravities would have equal differences; each $= \dfrac{w-m}{W+nm}$, w and m being constant. But $W+nm$

continually increases; and therefore $\frac{w-m}{W+nm}$ continually decreases.

26. Cor. 2. Hence the gradual diminution of the increments or decrements of specific gravity by equal additions of one ingredient to a constant measure of the other, does not, of itself, indicate a change of density of either of the ingredients; nor prove that in very diluted mixtures a greater proportion of one ingredient is absorbed, or lodged in the interstices of the other.

27. Cor. 3. Let $\delta = \frac{n.(w-m)}{W+nm}$, $\therefore m = \frac{nw - W\delta}{n.(1+\delta)}$.

Now w is the weight of the added ingredient; and therefore if δ, the variable part of the specific gravity be observed, it may be ascertained whether the magnitude of the added ingredient suffers any change.

28. It is frequently necessary to estimate the weights of bodies in troy ounces, and to refer their magnitudes to a cubic inch as a standard.

Now an oz. troy : an oz. avoirdupois :: 480 : 437.5;

$\therefore$ an oz. avoirdupois $= \frac{437.5}{480}$. oz. troy $= 0.911458$ oz. troy.

Also 1728 : 1 :: 1000 : the weight avoirdupois of a cubic inch of water, which therefore is $= 0.5787$ oz.; therefore the weight troy of a cubic inch

$$= 0.911458 \times 0.5787 = 0.52746;$$

whence, estimated in troy ounces, $W = 0.52746 \times MS$, or in troy grains $W = 253.1808 \times MS$; M being the magnitude in cubic inches, and the specific gravity of water being 1.

Prob. To determine the magnitude of a small irregular solid of known substance.

Let $S =$ its specific gravity, and let its weight in grains $= w$.

$$\therefore \text{ its magnitude in cubic inches } = \frac{w}{253.18 \times S}.$$

Prob. To determine the capacity of an irregular vessel.

Let the vessel be filled with water, and the weight of the water be (a) ounces, then $0.52746 : a :: 1 :$ the capacity of the vessel in cubic inches $= \dfrac{a}{0.52746}$.

Prob. To determine the diameter of any small sphere, whose specific gravity is s, its weight in grains being known.

The content of a sphere whose diameter is 1, $= 0.523598$; $\therefore 1 : 0.523598 :: 253.1808$ (grains in one cubic inch of water) : the weight of a globe of water whose diameter is one inch, which $\therefore = 132.5648$ grains. Now (11) $W \propto MS$, and spheres are as the cubes of their diameters, $\therefore w = 0.1325648 \times s\delta^3$, the specific gravity of water being 1,

$$\text{and } \delta = 0.19612 \sqrt[3]{\frac{w}{s}};$$

or if the specific gravity of water $= 1000$,

$$w = 132.5648 \times \frac{s\delta^3}{1000}, \text{ and } \delta = 1.9612 \sqrt[3]{\frac{w}{s}}.$$

Prob. To find the diameter of an uniform capillary tube.

Weigh the tube when empty, and when filled with mercury; and let $w =$ the difference of the weights in troy grains, $l =$ the length of the tube, and taking 13.568 for the specific gravity of mercury, a cubic inch will weigh 3435.16 grains very nearly, and

$$1 : \delta^2 l \times 0.785439 :: 3435.16 : w;$$

$$\text{whence } \delta = 0.019252 \sqrt{\frac{w}{l}}.$$

SECT. II.

29. EVERY particle of fluid presses equally in all directions.

For the establishment of this property we must have recourse to experiments; but it is the only case where they are necessary in order to explain the equilibrium of fluids.

(1) If in an empty vessel any number of tubes be placed, and mercury be poured into their lower orifices till it rises in the longer stems to a level with those orifices; upon filling the vessel with any fluid, it will be seen that the mercury is gradually pressed from the lower towards the higher orifices, which are above the surface of the fluid. Hence the angles, at which the stems are inclined, being of all degrees of magnitude, the pressure of the superincumbent fluid does not depend upon the direction in which it is exerted, that is, it is propagated in every direction.

(2) It is propagated *equally* in every direction. For, if any number of cylindrical tubes of equal diameters, each consisting of a longer and a shorter stem, inclined at different angles, be so immersed in a vessel of fluid that the longer stems may have their bases at the same perpendicular depth, and their other extremities above the surface, the fluid will be found to stand at the same altitude in each, viz. the altitude of the fluid in the vessel. Now the diameters of the tubes being the same, as also their altitude, the quantities of fluid supported must be the same in all the tubes; the causes of support therefore are the same; that is, the pressure of the fluid in the direction of the shorter stem is the same in each tube; therefore, independent of the angle formed by the stems, or equal in all directions.

(3) If an orifice be made in the surface of any vessel, either horizontal or inclined at any angle, a piston placed in it

to prevent the fluid from running out, would be pressed by the same force, if the perpendicular depth be the same.

30. This property constitutes a remarkable distinction between fluids and solids: the latter pressing with their whole weight in the direction of gravity alone.

31. Cor. Hence the lateral pressure of a fluid is equal to its perpendicular pressure.

This is one of the most extraordinary properties of fluids, and can be conceived to arise only from the extreme facility with which the component particles move amongst each other. It is not easy to imagine how this can take place, if the particles be supposed to be in immediate contact: they are therefore probably kept at a distance from each other by some repulsive force.

32. Every particle of a fluid at rest is pressed equally in all directions.

For if possible, let the particle be more pressed in one direction than in another: then, since the particles of a fluid yield to the smallest pressure, and are easily moveable amongst each other (1), motion will ensue in that direction in which the pressure is greatest, or the fluid will not be at rest, which is contrary to the supposition; therefore it is equally pressed in all directions.

33. Cor. 1. Hence appears the difference between the equilibrium of solids and fluids.

In *solid* bodies, if a force be applied at any point, it will, from the connexion of the parts, impel the whole mass in a direction parallel to that in which the force acts; and if to this force an equal one be applied in an opposite direction, there will be an equilibrium. But in *fluids,* if each drop taken separately be not pressed on every point of its surface equally in all directions, it will extend itself towards those parts where the pressure is least. Thus, if to a drop of fluid two equal forces be applied in opposite directions on opposite points of its surface, and each of these be represented by 1, and two equal forces, each represented by 2, be similarly applied at right angles to the former, the drop will not be in equilibrio; but will be lengthened on the parts where the forces 1 are applied, and flattened where the forces 2 are applied; and its particles will escape through the spaces between the points of application of the forces.

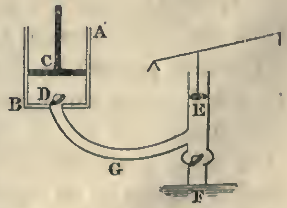
34. Cor. 2. If any number of pistons *A*, *B*, *C*, &c. be applied to orifices of different magnitudes in the sides of a vessel full of water, the forces acting on the pistons to maintain an equilibrium will be proportional to the respective orifices.

Since the pressure of every part of the piston *A* is communicated to the adjacent particles of fluid, and by them transmitted to every part of *B*, and *vice versâ*, it follows that these pressures will be in equilibrio, if they are equal. But the sum of the pressures propagated by *A* is proportional to the area of the orifice *A*; and the sum of the pressures propagated by *B* is proportional to the area of *B*; therefore there must be an equilibrium between these pressures when the force on *A* : the force on *B* :: the area of *A* : the area of *B*. The same is true whatever be the number of orifices.

35. Cor. 3. Hence by means of an incompressible fluid, we may, with a small power *P*, produce a pressure as great as we please, by taking the areas *A* and *B* in a proper ratio.

For $P : P' :: A : B, \ \therefore P' = P . \frac{B}{A}$.

36. Upon this principle Brahmah's Hydrostatic Press is constructed, by which the power of one part of a machine is communicated to another; or that of one machine to another, where circumstances prevent their being easily connected. This is done by means of forcing an incompressible fluid through a tube of small diameter into a larger in which is a



moveable piston made watertight. Thus let AB be a strong cylinder furnished with a solid piston C; into which is inserted a small tube DGE communicating with a small forcing-pump whose piston is E; and at D a valve opening upwards. The pressure against $C = \frac{C}{E} \times$ the pressure applied at E.

If the diameter of E be equal to the diameter of C, and a force of 10lbs. be applied to force E down, then C will be elevated with a force of 10lbs. If the diameter of E be half that of C, C will be raised with a force of 40lbs. And in general, if D and d be the diameters, and F the force at E, the force on $C = F \times \frac{D^2}{d^2}$. Let $d = 2$ inches, and $D = 24$, $F = 10$lbs. then the force on $C = 10 \cdot \overline{\frac{24}{2}\Big|}^{2} = 1440$lbs.

Since this force $\propto F \times \frac{D^2}{d^2}$, there is no limit to the power of the engine; for D may be increased and d diminished without limit.

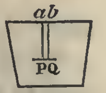
The force F may also be increased by lengthening the lever applied at E.

If the diameter of $C = 12$ inches, and of $E = \frac{1}{2}$ inch, and a force of 56lbs. could be applied downwards, which may be increased tenfold by the lever, the pressure on C will become 144 tons; and nearly all this is actually exerted. There is no friction except that occasioned by the pistons, which is small in comparison of the common cases of friction.

37. The pressure exerted by the particles of fluid at rest, against the surface of the vessel which contains it, is perpendicular to that surface.

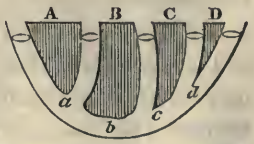
For if the pressure be not perpendicular to the surface, it cannot be destroyed by the reaction of the surface. Let it therefore be resolved into two, one perpendicular and the other parallel to the surface: the former will be destroyed by the reaction, and the latter, continuing to act on the particles of the fluid, will be transmitted in every direction; whence motion will ensue, which is contrary to the supposition; therefore the pressure must be perpendicular to the surface.

38. When a fluid is at rest, its surface is horizontal.



Let P and Q be two contiguous equal particles, equally distant from the horizon; if they are not equally distant from the upper surface of the fluid, the pressures on them in the direction of gravity being the weights of the incumbent columns Pa, Qb, will be unequal. Hence the pressures of these particles in opposite directions will be unequal, and motion will ensue which is contrary to the supposition. P and Q therefore are equally distant from the upper surface. And the same being true of every other two contiguous particles in the same horizontal plane, the upper surface must be horizontal.

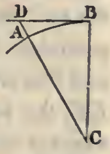
39. Cor. 1. If a communication be made between a fluid in one vessel, and a similar fluid in another, the surface of each will be at the same level, before the fluid is at rest: or if there be not sufficient to reduce them to a level, the whole will flow into the lowest.



40. Cor. 2. If into a vessel containing a quantity of fluid a solid be inserted, the surface will remain horizontal. Immerge, therefore, successively into the fluid the solid bodies Aa, Bb, Cc, Dd, &c. then after each immersion the surface will still be horizontal; and when all are immerged, the vessel will become a system of communicating vessels in which the surface of the fluid will be horizontal.

This is often demonstrated by supposing the parts Aa, Bb, Cc, Dd, &c. converted into ice without changing their former magnitude. When this happens, the equilibrium not being disturbed by supposing any part to become fixed, the fluid mass, whose surface was proved to be horizontal, will continue in the same state after the congelation of some of its parts; that is, the surface of the fluid in the communicating vessels will be horizontal.

41. Cor. 3. As the direction of gravity is in lines which are perpendicular to the surface of the earth, and it appears from the proposition that the surfaces of fluids are perpendicular to that direction, their surface will be a portion of a spheroid similar to that of the earth. When the surface is of no great extent, it may be considered as a plane: otherwise the curvature of the earth must be taken into the account.



Let BA be the arc of a circle whose centre is C, and let it represent a small portion of the curvature of the earth, all the points of it will cut the directions of gravity every where at right angles, that is, will be on a level. Draw the horizontal line BD to meet the vertical CA in D. This line will be the *apparent* level of B, and DA will be the *depression of the true level*. DA therefore must be taken from the observed height in order to obtain the real difference of the levels.

42. To determine AD.

$$\text{Let } CB = R,\ AB = a,$$

Since $DB^2 = DA.(DA + 2R)$, and AB being small compared with CB or CA, DA may be neglected compared with $2R$,

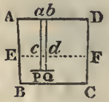
$$\therefore\ 2R \times DA = DB^2 = AB^2 \text{ very nearly, } = a^2$$

$$\text{and } DA = \frac{a^2}{2R}.$$

43. Cor. The elevation of the apparent above the true level is proportional to the square of the distance.

This supposes the visual ray to be a straight line: if greater accuracy be required, the effect of refraction must be taken into the account; since in consequence of the unequal densities of the air at different altitudes from the earth, the rays of light are continually bent out of their course. And from the mean of a number of experiments it has been inferred that the depression arising from the effect of refraction is $\frac{1}{7}$th of the elevation of the apparent above the true level. The correction, therefore, for the spherical figure of the earth and for refraction is $= \frac{6}{7} \cdot \frac{a^2}{2R}$.

44. The common surface of two fluids which do not mix, is horizontal.

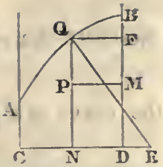


The upper surface AD of the lighter fluid (38) is horizontal. Let EF be the common surface of the two fluids, and P and Q two contiguous equal particles of the heavier fluid equally distant from the horizon, and therefore from AD: if they are not equally distant from EF, the vertical pressures on them will be unequal, being the weights of two columns containing different quantities Pc, Qd of the heavier, and ca, db of the lighter fluids; the pressures therefore in opposite directions will be unequal, and motion will ensue, contrary to the supposition. P and Q therefore are equally distant from EF; and the same being true of every other two contiguous particles in the same horizontal plane, EF will also be horizontal.

45. Cor. 1. The proposition is true whatever be the number of fluids: or the common surfaces are all horizontal. If therefore the number be infinite, or the density of the fluid vary according to any law, the surface of each will still be horizontal, and the equilibrium will not be disturbed.

46. Cor. 2. Hence the surfaces of fluids continue horizontal, when they are acted on by the pressure of the atmosphere.

Prob. To determine the nature of the curve which the surface of the fluid would assume, supposing it to be acted upon by gravity in the direction QN; and also by another force in the direction PM, which at every point P is proportional to PM its distance from a given line.



Suppose AB to be the surface of the fluid, and produce NP to meet it in Q. Let gravity $= g$, $DN = x$, $NQ = y$. The point Q is acted upon by g in the direction QN, and by a force $= my$ in the direction QF. Let QE be the compound force, and therefore perpendicular to the surface at Q.

$$\text{Now} \quad QN : NE :: g : my;$$

$$\text{or} \quad y : -\frac{y\,dy}{dx} :: g : my;$$

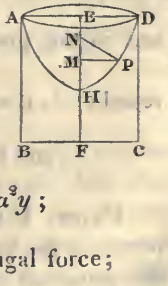
$$\therefore \frac{g}{m} . \frac{dy}{y} = -dx,$$

$$\text{and if } DB = a, \ \frac{g}{m}. \text{ hyp. log. } \frac{y}{a} = -x;$$

$$\text{or } y = ae^{-\frac{mx}{g}} \text{ the equation to the curve.}$$

Prob. If a cylindrical vessel containing any fluid be whirled round its axis perpendicular to the horizon with a given angular velocity; to determine the form which the surface of the fluid will assume.

Suppose $AHPD$ the form which the surface assumes. From any point P draw PM an ordinate to the axis EH, and PN the normal. Then P is acted upon by three forces, gravity in the direction NM, the centrifugal force in the direction MP, and the reaction of the fluid in the direction PN; and therefore the sides of the triangle PMN, taken in order, will represent these forces. Let α = the angular velocity, g = gravity;



$$\text{the centrifugal force} = \frac{v^2}{y} = \frac{a^2y^2}{y} = a^2y;$$

also, $NM : MP ::$ gravity : the centrifugal force;

or, $NM : y :: g : a^2y$;

therefore the subnormal $NM = \frac{g}{a^2}$, or is constant; whence the curve is a parabola whose latus rectum is $\frac{2g}{a^2}$.

Cor. If any number of fluids of different densities in the same cylindrical vessel are made to revolve round a vertical axis, their several surfaces will assume the form of paraboloids.

Prob. To determine the angular velocity so that a given quantity may be thrown out before there is an equilibrium.

$$\text{Let } ED = r, \text{ then } EH = \frac{a^2 r^2}{2g},$$

and the content $= \frac{1}{2}\pi r^2 . EH = \frac{\pi a^2 r^4}{4g} = Q$ by the supposition.

$$\text{Whence } a^2 = \frac{4Qg}{\pi r^4}, \quad \text{and } a = \frac{2}{r^2} . \sqrt{\frac{Qg}{\pi}}.$$

Prob. A vessel of given dimensions in the form of a paraboloid is filled with fluid, and revolves uniformly round an axis perpendicular to the horizon. To determine the time of rotation that it may just be emptied.

The surface forms a paraboloid such that

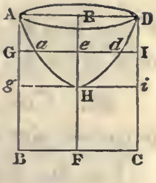
$NM : MP :: g : a^2 y$ where a is the angular velocity,

or $\frac{1}{2} l : y :: g : a^2 y$; $\therefore a^2 = \frac{2g}{l}$,

and in the present case $l =$ the latus rectum of the given vessel; $\therefore a = \sqrt{\frac{2g}{l}}$,

$$\text{and the time of revolution} = 360 \sqrt{\frac{l}{2g}}.$$

Prob. If the revolving cylindrical vessel be not full, the vertex of the paraboloid will sink as much below the level of the quiescent surface, as the water at the highest point rises above it.



Let GI be the level of the quiescent surface, and AHD the paraboloid formed by the revolution. Then since no water is thrown out, $AGa + DId = aHd$.

Let $He = x$, $AE = r$, $EH = h$, and the latus rectum $= l$; $\therefore\ ea^2 = lx$,

and the solid $aHd = \frac{1}{2}\pi lx^2$, and $AHD = \frac{1}{2}\pi hr^2$,

$$\text{and}\quad AGID = \pi r^2 . (h - x);$$

$$\text{but}\quad AadD = \tfrac{1}{2}\pi . (r^2 h - lx^2);$$

$$\therefore\ AGa + DId = \tfrac{1}{2}\pi r^2 h - \pi r^2 x + \tfrac{1}{2}\pi l x^2, \text{ which is } = \tfrac{1}{2}\pi l x^2;$$

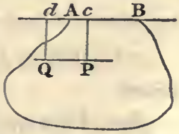
$$\text{whence}\quad \pi r^2 x = \tfrac{1}{2}\pi r^2 h;$$

$$\text{and}\quad x = \tfrac{1}{2}h.$$

47. Particles at equal perpendicular depths are equally pressed, when the fluid is at rest.

Let P and Q be two contiguous equal particles at equal perpendicular depths, or in the same horizontal plane; then if they are not equally affected by the surrounding medium, the pressures on them will be unequal: whence their pressures in opposite directions will be unequal, and motion will ensue, contrary to the supposition; P and Q therefore are equally pressed.

48. The pressure on a particle, wherever situated, is equal to the weight of a line of fluid particles whose length is the perpendicular depth of the particle pressed.



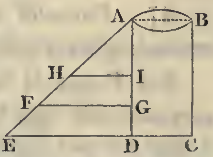
If the particle P be situated immediately below the upper surface AB of the fluid, the pressure upon it is manifestly equal to the weight of the incumbent line of fluid particles cP. And if Q be any other particle in the same horizontal plane, and Qd be drawn perpendicular to that plane, meeting BA produced in d; then since (47) particles at equal perpendicular depths are equally pressed, the pressure on Q is equal to that on P, that is, (since $Qd = Pc$) equal to the weight of a line of fluid particles, whose length is equal to the perpendicular depth of the particle pressed*.

49. Cor. 1. The pressure on any particle varies as its perpendicular depth.

For the fluid being of uniform density, the weight of a line of fluid particles will be proportional to its length (14).

50. Cor. 2. In any vessel, those parts that are deepest sustain the greatest pressure.

In forming vertical pipes, therefore, to convey water or any other fluid, and in constructing banks, in order that each part may be equally able to support the pressure against it, it is necessary that the lower parts should be the stronger.



If *ABCD* be a vessel of water, and there be taken, in the base produced, *DE* to represent the pressure at the bottom; joining *AE*, and drawing *GF*, *IH*, parallel to the base; *FG* will represent the pressure at the depth *AG*, and *HI* the pressure at the depth *AI* (49).

51. If the sides of a vessel be perpendicular, and its base parallel to the horizon, the pressure of the fluid on its base is the whole weight of the fluid.

The whole pressure of the fluid is sustained by the base and sides together; and the sides, being in the direction of

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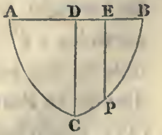
* If *ACB* be a curve supporting a fluid, its nature may be determined.



Let $DE=x$, $EP=y$, $CP=z$. Then since the pressure of the fluid is as its perpendicular depth, the pressure perpendicular to the curve $P \propto y$. This therefore is proportional to the curvature, or inversely as the radius of curvature; if therefore dz be constant, $y \propto \frac{d^2x}{dy\,dz}$; from which the nature of the curve may be determined.

gravity, sustain no part of that which is exerted perpendicularly downwards; the whole weight therefore of the fluid is sustained by the base; that is, the pressure on the base is equal to the weight of the fluid.

52. COR. 1. If $A =$ the area of the base, z the depth, and s the specific gravity; at a given place the pressure on the base may be represented by sAz. If therefore different vessels of this description be filled with different fluids, the pressures on the bases are as the areas of the bases, the perpendicular depths and the specific gravities of the fluids jointly.

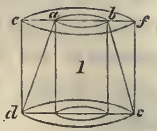
53. COR. 2. At any other place, where the force of gravity is altered in the ratio of $g : 1$, the pressure on the base will be $= sgAz$ (8).

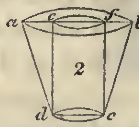
54. COR. 3. If $R =$ the radius of the earth, and g and g' be the forces of gravity at any altitudes a and a', $\frac{g'}{g} = \frac{(R+a)^2}{(R+a')^2}$; and a column of fluid whose length is l, observed at an altitude a' would exert a pressure equal to that exerted by one whose length is $l \cdot \left(\frac{R+a}{R+a'}\right)^2$ observed at an altitude a; or since the distances from the surface are small, at which observations can be made, compared with R, the third terms in the expansion may be neglected, and the length will be $l \cdot \left(1 + \frac{2a}{R}\right) \cdot \left(1 - \frac{2a'}{R}\right)$,

$$\text{or } l \cdot \left\{1 + \frac{2 \cdot (a - a')}{R}\right\} \text{ nearly.}$$

55. If the sides of a vessel be not perpendicular to the horizon, the pressure on the base, supposing it horizontal, may be greater or less than the weight of the fluid in the vessel.

The pressure on a particle of fluid P contiguous to the base is (48) the weight of a line of fluid particles whose length is the depth of P: and since every particle presses in all directions with the same force with which it is pressed, the pressure on the base is equal to the weight of such a column. And as this may be proved for every point in the base, the whole pressure upon it is equal to the weight of a column whose base is the base of the vessel, and altitude the perpendicular depth.





Now if the sides of the vessel *abcd* converge from the extremities of the base towards each other, it is manifest that the weight of such a column of fluid *edcf* will be greater than the weight of the fluid in the vessel. But if the sides diverge, the weight of the column *edcf* will be less than the weight of the fluid in the vessel. The pressure therefore on the base will be greater or less than the weight of the fluid in the vessel, according as the sides converge or diverge from the extremities of the base.

If the sides of the vessel converge, and therefore the pressure on its base be greater than the weight of the fluid, the additional pressure arises from the re-action of the sides. If the sides diverge, and the pressure on the base be therefore less than the weight of the fluid, the remaining weight is sustained by the sides.

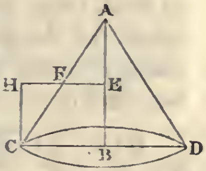
Ex. The pressure against the base of a cone, when the base is downwards, is the same as that of a cylinder of the same base and altitude: and as the content of the cylinder is treble that of the cone, the upward pressure against the sides of the cone is $\frac{2}{3}$ of the weight of such a cylinder; and the base of the cone sustains a pressure equal to three times the weight of the fluid which it contains.

56. It may be proper here to remark, with respect to the pressures upon the horizontal bases of vessels, that it is necessary to distinguish between the pressure which the plane *cd*, (see Fig. above) the base of the vessel *abcd*, would

sustain from the fluid only, and that which it sustains in supporting the vessel and the fluid it contains. If the bottom *cd* were detached from the vessel, to prevent the escape of the water it must be pressed upwards with a force equal to the weight of the cylinder *edcf* of the fluid: but in order to support the vessel, a force will be required equal to the weight both of the vessel and the fluid it contains. Thus when the vessel is narrowest at the base, it will require more force to support the vessel, than to keep its base from falling: but if widest at the base, less.

Prob. A hollow cone resting with its base on a horizontal plane, water is poured in at the top. How high will the water rise, before it raises the cone, and by that means escapes?

Let E be the point to which it rises just before it lifts the cone. Now when the equilibrium is just maintained, the weight of the conical shell is equal to the action of the water against the sides, that is, is equal to the re-action of FC, or to the weight of water in HFC.



Let $AB=a$, $BC=b$, $AE=x$. The solid content of the cone $= \frac{1}{3}\pi b^2 a$, and of the cone $AEF = \frac{1}{3}\pi b^2 a \cdot \frac{x^3}{a^3}$;

$$\therefore \text{ the content of the frustum } BEFC = \tfrac{1}{3} \cdot \pi b^2 a \cdot \left\{1 - \frac{x^3}{a^3}\right\},$$

$$\text{and the weight of water in } BEFC = \tfrac{1}{3} \cdot \pi b^2 a \cdot \left\{1 - \frac{x^3}{a^3}\right\},$$

if the specific gravity of water = 1.

Let m = the thickness of the conical shell, and s its specific gravity,

then the surface of the cone $= \pi b\sqrt{a^2+b^2}$,

and the weight of the cone $= ms\pi b\sqrt{a^2+b^2}$,

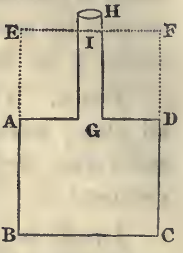
and the content of the cylinder $BH = \pi b^2 \cdot (a-x)$;

$$\therefore\ ms\pi b\sqrt{a^2+b^2}=\pi b^2\,.\,(a-x)-\tfrac{1}{3}\pi b^2 a\,.\left(1-\frac{x^3}{a^3}\right),$$

whence by the solution of a cubic equation x may be found.

57. Upon these principles also is explained the Hydrostatical Paradox, that the pressure upon the bottoms of vessels filled with fluid, does not depend upon the quantity of fluid which they contain, but solely upon the altitude.

If into the upper surface of any vessel *ABCD*, a tube be inserted, and the vessel filled with fluid to the point *I*, the pressure upon every point in the base will be the same as if the vessel were continued to *E* and *F*, and filled with fluid to *EF*. The pressure of the fluid therefore on the base of a vessel may be very great, whilst the weight of the fluid is small: and the addition of a small quantity in the tube *GH* may increase the pressure on the base *BC* in a very great proportion.



Prob. If the height of a cylinder be a foot, its diameter 6 inches; the diameter of a pipe fixed to the top of it one inch; and the cylinder and pipe be filled till the pressure on the bottom is to the weight of the fluid :: 9 : 1; to find the length of the pipe.

Let $x=$ its length, the weight of the fluid $=\pi s\,.\,\{9\times 12+x\,.\,\tfrac{1}{4}\}$

and the pressure on the base $=\pi s\,.\,9\,.\,\{x+12\}$;

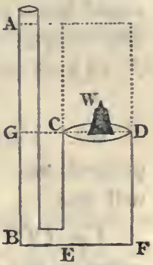
$$\therefore\ 9\,.\,(x+12)\ :\ 9\times 12+x\times\tfrac{1}{4}\ ::\ 9\ :\ 1,$$

$$\text{and}\quad x+12=108+\frac{x}{4};$$

$$\therefore\ x=128\text{ inches.}$$

58. The same principle will illustrate the Hydrostatic Bellows.

This instrument consists of a tube of small diameter about three feet high, communicating with a cylindrical vessel whose sides are made of leather like a pair of bellows, the upper and lower surfaces being formed of circular or oval boards. When a fluid is poured into the tube, it flows into the bellows and keeps the boards separated. Heavy weights are then placed upon the upper board; and by pouring more fluid into the tube, the moveable board with its load will be raised and kept in equilibrio by the column of fluid.



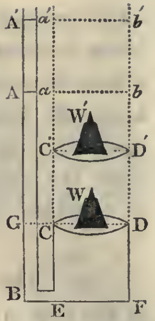
59. The weight of the supporting column in the tube *AG* is to the weight upon the upper board as the area of the section of the pipe to the area of the board.

For the fluid at *B*, the bottom of the pipe, is pressed by a force varying as the altitude *AB*; this pressure is communicated horizontally to all the particles in *EF*, and thence transmitted through the fluid in the bellows; the pressure therefore of the fluid upwards is equal to the weight of a cylinder of fluid whose base is *EF* and altitude *AB*; and the actual weight of the water supported is that of a cylinder whose base is *FE*, and altitude *BG*. The weight therefore which maintains the equilibrium will be that of a cylinder of fluid whose base is the same and altitude *AG*. Hence (14), the weight of the fluid in *AG* : *W* :: the area of a section of the pipe : the area *EF*.

60. Cor. 1. Hence a given quantity of fluid may be made to balance another quantity however great.

61. Cor. 2. If there be an equilibrium and more fluid be poured in, it will rise equally in the pipe and the other part of the machine.

Let *C* and *A* be the altitudes to which the fluid rises when the machine is in equilibrio; take $Ea = BA$, and draw the plane ab parallel to *CD*. Then the weight *W* is equal to the weight of a column of fluid $CabD$ (59). Suppose more fluid to be poured in, and let C' and A' be corresponding points to which it rises; take $Ea' = BA'$, and drawing the plane $a'b'$ parallel to $C'D'$, the weight W will be equal to the weight of a column of the fluid $C'a'b'D'$ (59). Hence, $CabD = C'a'b'D'$, and they have equal bases; $\therefore Ca = C'a'$, and consequently $CC' = aa'$. But $BA = Ea$, and $BA' = Ea'$; $\therefore AA' = aa'$, and consequently $CC' = AA'$.



62. COR. 3. Supposing a given quantity of fluid to be poured into the tube; to determine how much the weight will rise.

Let CC' be the altitude through which it rises, and the water in the tube to rise from A to A'. Let x and y represent the areas of the sections of the tube and bellows, and let the quantity poured into the tube $= lx$.

Let $z = CC' = AA'$,

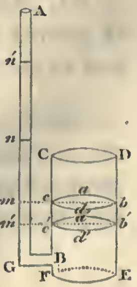
the content of the fluid column $C'D = yz$,

and the content of the column $A'A = xz$;

$$\therefore xz + yz = lx,$$

$$\text{and } z = \frac{lx}{x+y}.$$

63. If in the side of an open vessel a bent tube be inserted, and on the surface of the fluid a moveable lid exactly fitting the vessel be placed with a weight upon it, and the tube be graduated; any additional weight placed on the lid may be determined from knowing the height to which the fluid rises in the tube; and the converse.



Let $CDEF$ be a vessel whose upper part is open, and filled with fluid to the altitude Fc. On $abcd$, the surface of the fluid, let there be placed a moveable lid exactly fitting the vessel: let m be the level in the tube, which is inserted at B; and by the action of the weight $abcd$, and any other given weight placed upon it, let the water in the tube rise to some point n above m; AG being vertical, and the top A open.

Let $nm = h$, $s =$ the specific gravity of the fluid, $W =$ the weight on the surface of the fluid, and $A =$ the area of the surface, $a =$ the horizontal section of the tube. The pressure of the fluid at $m = sha$ (52), and since there is an equilibrium with W on A, $sha : W :: a : A$ (59);

$$\therefore\ W = shA.$$

Now on the addition of any weight W' (to be determined) suppose the fluid to rise to n', and the level of the fluid in the vessel to be depressed to $a'b'c'd'$, or to m'.

$$\text{Let } nn' = x,\ mm' = y,$$
$$\text{then } W + W' = sA.(x + y + h).$$

But the fluid being supposed incompressible, the descent y and the elevation x will be inversely as the areas A and a, or $Ay = ax$; whence

$$W + W' = s.\{(A + a).x + Ah\}.$$
$$\text{But } W = sAh;$$
$$\therefore\ W' = sx.(A + a).$$

If therefore x the elevation above the first level be known, the additional weight W' may be determined: and the converse.

64. If a plane be immersed in a fluid, the pressure perpendicular to its surface is equal to the weight of a column of fluid, whose base is the area of the plane, and altitude the perpendicular depth of its centre of gravity; the fluid being at rest, and acted on only by gravity.

Suppose the surface P, divided into an indefinite number of portions A, B, C, &c. so small that every point of any one of them may be considered as at the same perpendicular depth below the surface of the fluid: and let their respective perpendicular depths be a, b, c, &c. Then the pressure of the fluid perpendicularly against any one of them A, is measured by the number of fluid particles contiguous to it, that is, by A, and the pressure of each jointly (52); that is, by sAa, s being the specific gravity of the fluid. In the same way the pressure on B is sBb, &c.; therefore the sum of the pressures, or the whole pressure perpendicular to the plane

$$= s\,.\{Aa + Bb + Cc + \&\text{c.}\}$$

But A, B, C, &c. may be considered as bodies, and the surface of the fluid as a plane given in position;

$$\therefore\ Aa + Bb + Cc + \&\text{c.} = (A + B + C + \&\text{c.})\,.\,G,$$

if G be the depth of the centre of gravity*.

Hence, the pressure perpendicular to the surface

$$= (A + B + C + \&\text{c.})\,.\,Gs = PGs,$$

which is the weight of a column of fluid, whose base is P, altitude G, and specific gravity s. (22).

65. Cor. 1. Hence, the pressures vary as the areas of the planes, the perpendicular depths of their centres of gravity, and the specific gravities of the fluid jointly. And if different planes be immersed in the same fluid, the pressures perpendicular to their surfaces, are as their areas, and the perpendicular depths of their centres of gravity.

66. Cor. 2. Hence, the pressure on a plane surface of given dimensions, if it be parallel to the surface of the fluid, varies as its perpendicular depth: if inclined, as the perpendicular depth of its centre of gravity.

67. Cor. 3. Hence, the perpendicular pressure on all planes of equal areas, whatever be their figure, immersed in the same fluid, will be the same, provided the depths of their centres of gravity remain the same. And if any plane revolve round its centre of gravity, which remains fixed, the pressure perpendicular to its surface will remain the same as when it was horizontal.

68. Cor. 4. The whole pressure on the sides of a vessel, which are perpendicular to its base is equal to the

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* Wood's *Mechanics*, 172. Whewell's *Mechanics*, 61.

weight of a rectangular prism of the fluid, whose altitude is that of the fluid and base a parallelogram, one side of which is equal to the altitude of the fluid, and the other to half the perimeter of the vessel.

69. Cor. 5. If the perpendicular pressures upon a given area immersed in two different fluids be equal, the perpendicular depths of their centres of gravity will vary inversely as the specific gravities of the fluids.

If therefore s and s' represent the specific gravities of two fluids, and z the depth at which a plane is immersed in the first, the depth at which it must be immersed in the second to sustain the same pressure $= z \times \frac{s}{s'}$.

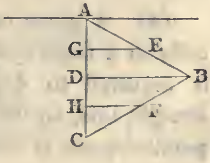
(1) To compare the pressures upon two physical lines *just perpendicularly* immersed in a fluid.

Let the lines be represented by A and B; then since the centre of gravity of a physical line is its middle point, the pressure on A : the pressure on B

$$:: A \times \frac{A}{2} : B \times \frac{B}{2} :: A^2 : B^2.$$

Cor. If the lines be inclined at given angles to the surface, the perpendicular depth of the centre of gravity of each will be equal to half the line × sine of inclination; the pressures therefore will be as the squares of the lines, and the sines of inclination jointly.

(2) To compare the pressures on the three sides of an equilateral triangle just immersed in a fluid in such a manner that one side may be perpendicular to the surface of the fluid.



Let AC be perpendicular to the surface. From E and F the points of bisection, and therefore the centres of gravity of AB, BC, draw EG, FH perpendicular to AC, to which also let BD

be perpendicular; then it is evident that $AD = CD$, or D is the centre of gravity of AC.

Now (Euclid VI. 2.) $AG : AD :: AE : AB :: 1 : 2$,

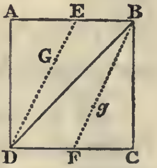
and $DH : CD :: 1 : 2$;

$\therefore AH : (CD =) AD :: 3 : 2$,

and AG, AD, AH the perpendicular depths of the centres of gravity of the sides are as 1, 2, 3. Hence (66), the pressures are in the proportion of the numbers 1, 2, 3.

Cor. The pressure on BC is equal to the sum of the pressures on AB and AC.

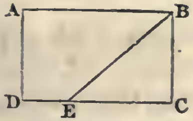
(3) A square $ABCD$ is immersed vertically in a fluid, the side AB coinciding with the surface; if the diagonal BD be drawn, compare the pressures on the triangles ABD, BDC.



Bisect AB, DC in E and F. Join DE, BF; take $EG = \frac{1}{3} ED$, and $Fg = \frac{1}{3} BF$; G and g are the centres of gravity of the triangles ABD and BDC; and their perpendicular depths are as $EG : Bg :: \frac{1}{3} ED : \frac{2}{3} BF :: 1 : 2$; therefore (66), the pressures on ABD and BDC are as $1 : 2$.

Cor. The same is true in the case of a rectangle. And the proportions in this and the following problems will be true, whatever be the inclinations of the immersed plane, provided only that AB coincides with the surface: for the perpendicular depths of the centres of gravity will be altered in a given ratio.

(4) A given rectangular parallelogram is immersed vertically in a fluid with one side coincident with the surface. It is required to draw from one of the angles to the base a straight line, so that the pressures on the parts into which the parallelogram is divided, may be in a given ratio of $m : n$.



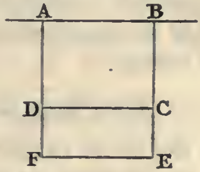
Suppose BE the line; then the pressure on the whole : the pressure on BEC

$$:: m+n : n,$$

or $AB \times AD \times \frac{1}{2}AD : \frac{1}{2}CE \times CB \times \frac{2}{3}CB :: m+n : n$;

$$\therefore CE = \frac{3}{2} \cdot \frac{n}{m+n} \cdot AB.$$

Cor. Let $CE = CD = AB$, and $\therefore 2m = n$, the pressure on the upper : pressure on the lower part :: 1 : 2, as before.



(5) To compare the pressures on two rectangles $ABCD$, $DCEF$, of the same breadth, immersed vertically in a fluid, whose surface is AB.

The pressure on $ABCD$: that on $ABEF$

$$:: AD \times \tfrac{1}{2}AD : AF \times \tfrac{1}{2}AF$$

$$:: AD^2 : AF^2;$$

$\therefore$ the pressure on $ABCD$: that on $DCEF$ :: $AD^2 : AF^2 - AD^2$.

Cor. 1. If $AF : AD :: \sqrt{2} : 1$, the pressures are equal.

Cor. 2. To draw a line parallel to AB, so that the pressures on the upper and lower parts of the rectangle $ABEF$ may be as $m : n$.

$$AD^2 : AF^2 - AD^2 :: m : n;$$

$$\therefore AD^2 : AF^2 :: m : m+n,$$

$$\text{and } AD = AF \cdot \sqrt{\frac{m}{m+n}}.$$

Cor. 3. To divide the rectangle into n parts so that the pressures on each may be equal.

Let DE be the lowest, then $AF^2 : AD^2 :: n : n-1$,

$$\text{and } AD = \sqrt{\frac{n-1}{n}} \cdot AF;$$

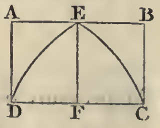
whence $DF = AF.\left\{1 - \sqrt{\frac{n-1}{n}}\right\} = AF.\frac{\sqrt{n}-\sqrt{n-1}}{\sqrt{n}}$, the lowest section.

For n substitute $n-1$, and the height of the next will be determined; and so on.

Cor. 4. From this problem we might determine a limit to the requisite thickness of flood-gates, and their strength according to the stress on them. For the pressure on AC : that on AE :: $AD^2 : AF^2$; and therefore the thickness should be as the square of the depth, and decrease from the bottom to the surface.

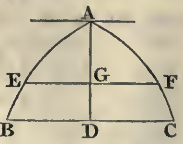
(6) Compare the pressure on the area of a parabola with that on its circumscribing rectangle, both being immersed perpendicularly, to the vertex.

Since the areas of the parabola and rectangle are as $2 : 3$; and the depths of their centres of gravity as $\frac{3}{5}EF : \frac{1}{2}AD$; therefore (65) the pressures on them are as $4 : 5$.



(7) If a parabola be just immersed vertically in a fluid; at what distance from the vertex must a line be drawn parallel to the base, so that the pressure on the upper part may be to that on the lower :: $m : n$.

The pressure on the upper part : the pressure upon the whole :: $m : m+n$;



$$\therefore AG \times GE \times AG : AD \times DB \times AD :: m : m+n,$$

$$\text{or } AG^{\frac{5}{2}} : AD^{\frac{5}{2}} :: m : m+n;$$

$$\therefore AG = AD.\left(\frac{m}{m+n}\right)^{\frac{2}{5}}.$$

(8) The pressure on the four sides of a cube filled with a fluid, is double the weight of the contained fluid.

The areas of the base and a side being equal, and the depths of their centres of gravity :: 2 : 1, (66) the pressure on the base is to that on a side :: 2 : 1; therefore the pressure on the base is to that on the four sides :: 2 : 4 :: 1 : 2, or, (51) the weight of the fluid is to the pressure on the four sides :: 1 : 2.

Cor. The pressure on the sides and base is equal to three times the weight of the fluid.

(9) If on the side of a vessel a number of circles be described, the pressures on which are proportional to their diameters: the ratio of their distances from the surface may be found.

For the pressure $\propto$ area $\times$ depth of centre of gravity (65)

$$\propto R^2 \times D.$$

But by the supposition it $\propto R$;

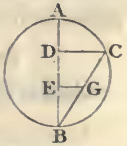
$$\therefore\ R^2 \times D \propto R,$$

$$\text{and } D \propto \frac{1}{R}.$$

Cor. If the pressure $\propto R^n$, $D \propto R^{n-2}$.

(10) A circle being just immersed vertically in a fluid; draw from the lowest point that chord on which the pressure shall be the greatest.

From B the lowest point, draw the vertical diameter BA, and let BC be the chord, which bisect in G; and draw GE, CD perpendicular to AB.



Let $BE = x$; $\therefore\ BD = 2x$, $BC = \sqrt{4rx}$, and $AE = 2r - x$;

whence (65), $(2r - x)\sqrt{4rx} = \text{max.}$

$$\text{or } 2rx^{\frac{1}{2}} - x^{\frac{3}{2}} = \text{max.}$$

$$\therefore\ rx^{-\frac{1}{2}}dx - \tfrac{3}{2}x^{\frac{1}{2}}dx = 0.$$

$$\text{and } x = \tfrac{2}{3}r.$$

(11) If two spheres be just immersed in any fluid; to compare the pressures upon them.

Since (65) the pressures are as the surfaces × the depths of their centres of gravity, they are as $R^2 \times R \propto R^3$.

(12) If a hollow sphere be filled with fluid; the whole pressure against the internal surface is three times the weight of the contained fluid.

Let $r =$ its radius; $\therefore 4\pi r^2 =$ its internal surface, and if $s =$ the specific gravity of the fluid, the pressure

$$= s \,.\, r \,.\, 4\pi r^2 = 4\pi r^3 s;$$

but the content of the sphere $= \frac{4}{3}\pi r^3$, and its weight $= \frac{4}{3}\pi r^3 s$;
$\therefore$ the pressure : the weight of the fluid $:: 4\pi r^3 s : \frac{4}{3}\pi r^3 s :: 3 : 1$.

(13) A tetrahedron is filled with fluid. Compare the pressure on the base with the pressure on the sides, and the weight of the fluid.

If B and P represent the base and perpendicular altitude, the pressure on the base : that on a side $:: B \times P :$ side $\times \frac{2}{3}P$

$$:: 3 : 2;$$

therefore the pressure on the three sides is double the pressure on the base. Also

the pressure on the base : the weight of the fluid

$:: B \times P :$ solid content of the fluid

$:: 3 : 1;$

therefore the whole pressure on the sides, the pressure on the base, and the weight of the fluid are as 6, 3, 1.

(14) To compare the pressure against the bottom of a cylinder with that against its sides.

If $a =$ the altitude of the cylinder, and $r =$ the radius of the base, the pressure on the bottom $= \pi a r^2 s$, and the pressure on the sides $= \frac{1}{2}a \,.\, 2\pi r a s = \pi r a^2 s$;

$$\therefore \text{the pressures are as } \pi a r^2 s : \pi r a^2 s :: r : a.$$

Cor. The whole pressure sustained by the cylinder

$$= \pi a r s \,.\, \{r + a\}.$$

(15) The concave surface of a cylinder filled with fluid is divided by horizontal sections into n annuli, in such a manner that the pressure on each annulus is equal to the pressure upon the base. Given the radius of the cylinder, to find its height; and also the breadth of the (p^{th}) annulus.

Let h = the height, and r = the radius of the base, and a = the breadth of the first annulus,

$$\text{then } 2\pi r a \times \frac{a}{2} = \pi r^2 h;$$

$$\therefore a = \sqrt{rh}.$$

Let a' = the breadth of the second,

$$\text{then } 2\pi r a' \times \left(a + \frac{a'}{2}\right) = \pi r^2 h;$$

$$\therefore 2a'\sqrt{rh} + a'^2 = rh,$$

$$\text{and } a'^2 + 2a'\sqrt{rh} + rh = 2rh;$$

$$\therefore a' = (\sqrt{2} - 1).\sqrt{rh}.$$

Let a'' = the breadth of the third;

$$\therefore 2a''.\left\{a + a' + \frac{a''}{2}\right\} = rh,$$

$$\text{or } 2a''.\left\{(1 + \sqrt{2} - 1).\sqrt{rh} + \frac{a''}{2}\right\} = rh;$$

$$\text{that is, } a''^2 + \sqrt{2rh}.2a'' = rh;$$

$$\text{whence } a'' = \{\sqrt{3} - \sqrt{2}\}.\sqrt{rh}.$$

In the same way, the breadth of the fourth annulus

$$= (\sqrt{4} - \sqrt{3}).\sqrt{rh};$$

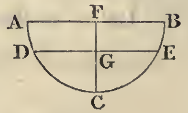
and the breadth of the (p^{th}) $= (\sqrt{p} - \sqrt{p-1}).\sqrt{rh}$; therefore the sum of the breadths or $h = \sqrt{n}.\sqrt{rh}$;

$$\text{and } h^2 = nrh; \ \therefore h = nr.$$

(16) A cylinder has some fluid in it. Suppose from a change of temperature the bulk of the fluid to be increased $\frac{1}{n}^{\text{th}}$ part; what alteration will take place in the pressures on the sides and base?

Since (65) the pressure ∝ area × perpendicular depth of its centre of gravity × the specific gravity of the fluid; and the part pressed of the sides is *increased* in the ratio of $n : n+1$; therefore the altitude, and therefore also the depth of the centre of gravity is *increased* in the ratio of $n : n+1$; but (13) the specific gravity is *diminished* in the ratio of $n+1 : n$; whence, compounding these proportions, the pressure on the sides will be *increased* in the ratio of $n : n+1$. And the pressure on the base, which is equal to the weight of the fluid, remains the same.

(17) A semi-circle is immersed vertically in a fluid, with the diameter contiguous to the upper surface; to determine on which of the chords, parallel to the surface, the pressure is the greatest, supposing the density of the fluid to increase as the depth.



Let DE be the chord, then the pressure $\propto DG \times GF^2$

$$\propto \sin DC \times \cos.^2 DC = \text{max.}$$

$$\therefore \sin z \times \cos^2 z = \text{max.}$$

$$\text{and } dz \,.\, \cos.^3 z - 2dz \,.\, \sin^2 z \,.\, \cos z = 0;$$

$$\text{or } \cos^2 z = 2 \sin^2 z = 2 - 2\cos^2 z;$$

$$\text{and } \cos . z = \sqrt{\tfrac{2}{3}}; \quad \therefore FG = \sqrt{\tfrac{2}{3}} \,.\, FC.$$

Cor. 1. If the figure be a parabola, and $FC = b$, $FG = x$,

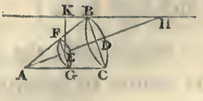
$$\text{and latus rectum } = 4a, \ DE = 4\sqrt{a.(b-x)};$$

$$\therefore \sqrt{b-x} \,.\, x^2 = \text{max.}$$

$$\text{or } bx^4 - x^5 = \text{max.} \quad \text{whence } x = \tfrac{4}{5} b.$$

Cor. 2. If the figure be a triangle, $FG = \frac{2}{3} FC$.

(18) A given cone filled with fluid is supported with its axis inclined to the horizon at a given angle. On what section parallel to the base is the pressure a maximum?



Let ABC be the cone with its axis AD inclined to the horizon at an angle α, and FEG the section required. Let

BH be horizontal and EK vertical; $AD = a$, $BD = b$,

$$\therefore\ AH = \frac{b}{\tan a} + a = c,\ AE = x, \text{ and } HE = c - x;$$

$$\therefore\ EK = (c - x) \,.\, \sin.\, a;$$

$$\text{and } FE^2 = \frac{b^2}{a^2} \,.\, x^2; \ \therefore \text{ the circle } FEG = \frac{\pi b^2}{a^2} \,.\, x^2,$$

$$\text{and (65) the pressure on } FEG \propto \frac{\pi b^2}{a^2} \,.\, x^2 \,.\, (c - x) \,.\, \sin a,$$

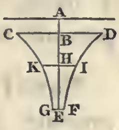
$$\propto x^2 \,.\, (c - x);$$

$$\therefore\ cx^2 - x^3 = \text{max.}$$

$$\text{and } 2cx dx - 3x^2 dx = 0;$$

$$\text{whence } x = \tfrac{2}{3} c = \tfrac{2}{3} AH.$$

(19) If a globe, whose radius at the bottom of the sea $= a$, ascend to the top, the depth being $= h$; what will be its diameter at the top, and what will be the locus of the extremity of its radius, the line in which the centre ascends being the abscissa.

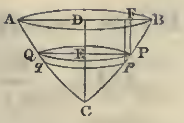


Let CBD be the surface of the water, EF the radius of the globe at the bottom; FID, GKC the curves described by the radii in ascending. Let $AB =$ the height of a column of water of the same weight as the atmosphere $= h'$, $EF = a$, $EB = h$, $BH = x$, $HI = y$. The magnitude of the globe being inversely as the pressure, $h + h' : h' + x :: y^3 : a^3$;

$$\therefore\ y = a \,.\, \sqrt[3]{\frac{h + h'}{h' + x}} \text{ the equation to the curve;}$$

$$\text{and at the surface, when } x = 0,\ y = a \,.\, \sqrt{\frac{h + h'}{h'}}.$$

The line drawn through A parallel to CD is an asymptote to the curve.



70. To find the pressure perpendicular to a surface of revolution.

Let C be the vertex of the surface, and CD its axis; $PQqp$ an annulus of the surface intercepted between two planes perpendicular to the axis, and indefinitely near to each other. Then each particle of this surface may be considered as equally distant from AB the surface of the fluid, and therefore equally pressed.

Let $CD = h$, $CE = x$, $EP = y$, $FP = DE = h - x$.

Then the pressure on the annulus = the annulus × its depth, (if the specific gravity of fluid = 1), $= 2\pi y\, d\Sigma . (h - x)$; whence the pressure on the curve surface

$$= 2\pi \int y\, d\Sigma . (h - x) + C.$$

Ex. 1. Suppose the surface to be a segment of a sphere with its vertex downwards.

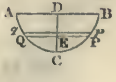
Let r = its radius, then $y\, d\Sigma = r\, dx$;

$\therefore$ the pressure $= 2\pi r . \int (h - x) . dx = 2\pi r . \{hx - \frac{1}{2} x^2\}$, C being $= 0$; and for the whole segment $x = h$, $\therefore$ the pressure $= \pi r h^2$.

Cor. 1. If the segment be a hemisphere, $h = r$, and the pressure $= \pi r^3$.

Cor. 2. If a whole sphere be taken, $h = 2r$, and the pressure $= 4\pi r^3$.

(2) Suppose the surface to be generated by a segment of a circle moving parallel to itself.



Let ACB be a section of the vessel made by a plane perpendicular to its axis, and let PQ, pq be ordinates indefinitely near to each other and parallel to AB. Let $CE = x$, $EP = y$, $CP = \Sigma$, $CD = h$, length of the vessel $= l$.

The pressure on $Pp = Pp \times DE$;

$\therefore$ pressure on Pp + pressure on $Qq = 2Pp \times DE$
$$= 2.(h-x).d\Sigma,$$

and this multiplied by l will give the pressure on an element of the vessel;

$$\therefore \text{ pressure on the vessel } = 2l.\int(h-x).d\Sigma$$

$$= 2l.\int(h-x).\frac{r\,dx}{\sqrt{2rx-x^2}},$$

$$= 2lr.\int\left\{\frac{h\,dx}{\sqrt{2rx-x^2}} - \frac{x\,dx}{\sqrt{2rx-x^2}}\right\}$$

$$= 2lr.\int\left\{(h-r)\frac{dx}{\sqrt{2rx-x^2}} + \frac{(r-x).dx}{\sqrt{2rx-x^2}}\right\}$$

$$= 2lr.\left\{(h-r).\text{arc}\left[\text{versin} = \frac{x}{r}\right] + \sqrt{2rx-x^2}\right\},$$

C being here $= 0$.

Let $x = h$, then the whole pressure

$$= 2lr.\left\{(h-r).\text{arc},\left[\text{versin} = \frac{h}{r}\right] + \sqrt{2rh-h^2}\right\}.$$

Cor. 1. Suppose the segment to be a semi-circle, then $h = r$, and the pressure $= 2lr\sqrt{2r^2 - r^2} = 2lr^2$.

Cor. 2. Suppose it to be a circle, then $h = 2r$, and

$$\text{the pressure } = 2lr.\left\{r.\text{arc},\left[\text{versin} = \frac{2r}{r}\right] + \sqrt{4r^2 - 4r^2}\right\}$$

$$= 2lr^2.\pi = 2\pi lr^2.$$

(3) To divide a hemispheroidical bowl, whose vertex is downwards, into two parts by a plane perpendicular to its axis, so that the pressure on the upper and lower parts may be equal.

Let $DE = x$, $EP = y$, $DC = a$, $DB = b$;

$$\therefore y = \frac{b}{a}.\sqrt{a^2 - x^2},$$

$$\text{and}\quad d\Sigma = dx\sqrt{1+\frac{dy^2}{dx^2}} = \frac{dx\sqrt{a^2-c^2x^2}}{\sqrt{a^2-x^2}},$$

$$\text{where } c^2 = \frac{a^2-b^2}{a^2};$$

$$\therefore \text{ the pressure} = 2\pi\int\frac{xy\,dx\sqrt{a^2-c^2x^2}}{\sqrt{a^2-x^2}} = \frac{2\pi b}{a}\cdot\int x\,dx\sqrt{a^2-c^2x^2}$$

$$= -\frac{2\pi b}{3ac^2}\cdot\{a^2-c^2x^2\}^{\frac{3}{2}} + C.$$

Now when $x=0$, pressure $=0$; $\therefore\ 0 = -\frac{2\pi b}{3ac^2}\cdot a^3 + C,$

whence the pressure on $AQPB = \frac{2}{3}\cdot\frac{\pi b}{ac^2}\cdot\{a^3-(a^2-c^2x^2)^{\frac{3}{2}}\}.$

Let $x = a$, then the pressure on ACB

$$= \frac{2}{3}\cdot\frac{\pi b}{ac^2}\cdot\{a^3-(a^2-a^2c^2)^{\frac{3}{2}}\}$$

$$= \frac{2}{3}\cdot\frac{\pi b}{ac^2}\cdot\{a^3-b^3\},$$

and the pressure on ACB : the pressure on $AQPB$

$:: a^3-b^3 : a^3-\{a^2-a^2x^2\}^{\frac{3}{2}} :: 2 : 1$, by the hypothesis;

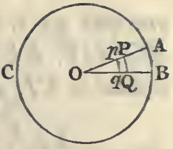
$$\therefore a^3-b^3 = 2a^3 - 2\,.\,(a^2-c^2x^2)^{\frac{3}{2}},$$

$$\text{or}\quad \tfrac{1}{2}\,.\,(a^3+b^3) = (a^2-c^2x^2)^{\frac{3}{2}};$$

$$\therefore x = \frac{1}{c}\cdot\sqrt{a^2-(\tfrac{1}{2}\cdot[a^3+b^3])^{\frac{2}{3}}}$$

(4) A cylinder of fluid revolves with a given velocity round its axis which is vertical. To find the pressure on its surface, (gravity being neglected).

Let ABC be a horizontal section, and let $h =$ the height of the cylinder, $OA = r$, $OP = x$, angle $AOB = \theta$, and angular velocity $= a$.



The velocity of P = angular velocity × distance OP, $= \alpha x$; therefore the accelerating force on $P = \frac{v^2}{\text{rad.}} = \frac{\alpha^2 x^2}{x} = \alpha^2 x$,
and the mass $PQqp = Pp \times PQ = \theta x \times dx$;
∴ the weight of $PQqp$ = mass × accelerating force $= \alpha^2 \theta x^2 dx$,
and the weight of $PQqp$: pressure which it exerts on AB

$$:: PQ : AB :: x : a;$$

$$\therefore \text{ the pressure of } PQqp \text{ on } AB = a\alpha^2 \theta x dx,$$

$$\text{and the pressure of } OPQ = \tfrac{1}{2} a \alpha^2 \theta x^2 + C;$$

$$\text{but } C = 0, \text{ since pressure} = 0, \text{ when } x = 0.$$

Let $x = a$, then

$$\text{the whole pressure of } OAB \text{ on } AB = \tfrac{1}{2} a^3 \alpha^2 \theta;$$

∴ the effect of the whole circle $= \frac{1}{2} a^3 \alpha^2 \times 2\pi = \pi \alpha^2 a^3$, and the pressure on the internal surface of the cylinder = pressure on $ACB \times h = \pi \alpha^2 a^3 h$.

Cor. The weight of the fluid $= \pi a^2 hg$;
∴ the pressure : the weight $:: \pi \alpha^2 a^3 h : \pi a^2 hg :: \alpha^2 a : g$.

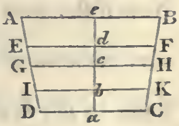
If $a = 1$ foot, and the time of revolution $= 1''$,

$$\alpha = \text{velocity} = \frac{\text{circumference}}{\text{time}} = 2\pi;$$

$$\therefore \text{ pressure } : \text{ the weight } :: 4\pi^2 : g.$$

In a similar manner it may be shewn, that if a cube be filled with fluid and revolve on an axis perpendicular to the surface, and passing through its centre; the pressure on the interior surface : the weight of the cube of fluid $:: 4\alpha^2 a : 3g$.

71. To find the pressure on the horizontal base of a vessel containing different fluids.



Let EF, GH, IK be the surfaces of the different fluids, these are horizontal (44); and let ab, bc, cd, de be the perpendicular altitudes of the fluids, whose specific gravities are s, s', s'', s''' respectively. Then the pressures of KG, GF, FA will be the same as those of portions of fluid similar to CI, and whose altitudes are

$$bc \,.\, \frac{s'}{s}, \quad cd \,.\, \frac{s''}{s}, \quad de \,.\, \frac{s'''}{s} \text{ respectively (69);}$$

$$\therefore \text{ the pressure on } DC = s \,.\, DC \,.\, \left\{ ab + bc \,.\, \frac{s'}{s} + cd \,.\, \frac{s''}{s} + de \,.\, \frac{s'''}{s} \right\}$$

$$= DC \,.\, \{ ab \,.\, s + bc \,.\, s' + cd \,.\, s'' + de \,.\, s''' \},$$

that is, it is equal to the base multiplied by the sum of the products of the specific gravity of each fluid into the perpendicular altitude of that fluid.

72. Cor. The pressure on

$$DC = DC \times s' \,.\, \left\{ ab \,.\, \frac{s}{s'} + bc + cd \,.\, \frac{s''}{s'} + de \,.\, \frac{s'''}{s'} \right\},$$

$$\text{or } = DC \times s'' \times \left\{ ab \,.\, \frac{s}{s''} + bc \,.\, \frac{s'}{s''} + cd + de \,.\, \frac{s'''}{s''} \right\}.$$

In the place therefore of a fluid of variable density one may be substituted whose density will be the same through the whole height.

(1) A cylindrical vessel whose height $= a$, and the radius of whose base $= r$, is filled with mercury to half its height, and the remainder is filled with water. Supposing the specific gravities to be as $14 : 1$, compare the pressure on the base with that on the concave surface.

The pressure on the base (71) is

$$= \pi r^2 \,.\, \{ \tfrac{1}{2} a \,.\, 14 + \tfrac{1}{2} a \,.\, 1 \} = \frac{15}{2} \,.\, \pi r^2 a.$$

Now if the water was removed, and in its place a column of mercury substituted whose altitude $= \frac{1}{28} \,.\, a$, the pressure on every point in the lower half of the vessel would not be altered (69), and it would be $= 2\pi r . \frac{1}{2} a . 14 . \left\{\frac{1}{4} a + \frac{1}{28} . a\right\}$

$$= 14 \pi r a . \frac{2}{7} a = 4 \pi r a^2,$$

and the pressure on the upper half $= 2\pi r . \frac{1}{2} a \times \frac{1}{4} a = \frac{1}{4} \pi r a^2$;

∴ the pressure on the whole concave surface

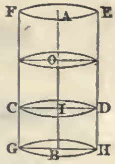
$$= 4 \pi r a^2 + \tfrac{1}{4} \pi r a^2 = \frac{17}{4} . \pi r a^2,$$

whence the pressure on the base : the pressure on the concave surface

$$:: \frac{15}{2} . \pi r^2 a : \frac{17}{4} . \pi r a^2$$

$$:: 30 r : 17 a.$$

(2) The specific gravities of two fluids which will not mix, are as $n : 1$. Compare the quantities which must be poured into a cylindrical vessel whose length is a inches, that the pressures on the concave surfaces of the tube which are in contact with the fluid, may be equal.



Let $AB = a$, $AI = x$, CD being the common surface of the fluids. To determine OI the altitude of a column of the heavier fluid, equal in weight to the column $CDEF$, let $n =$ the specific gravity of the lighter fluid; ∴ $OI = nx$ (69) and the pressure on $CDEF$: the pressure on $CGHD$

$$:: x . \frac{x}{2} . n : (a - x) . \{nx + \tfrac{1}{2}(a - x)\},$$

$$\text{whence } n x^2 = 2nx . (a - x) + (a - x)^2,$$

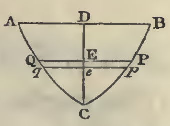
$$\text{and } (a - x)^2 + 2nx . (a - x) + n^2 x^2 = (n^2 + n) x^2;$$

$$\therefore a - x = \sqrt{n^2 + n} . x - nx,$$

$$\text{and } x = \frac{a}{1 - n + \sqrt{n^2 + n}},$$

$$\text{and } AI : IB :: 1 : \sqrt{n^2 + n} - n.$$

73. To determine the pressure against any part of the surface of a vessel, when the density varies according to any law.



Through any two points P, p indefinitely near to each other, draw PQ, pq perpendicular to the direction of gravity; then on every point in Pp or Qq there will be a perpendicular pressure equal to the pressure on E; as is evident from substituting (72 and 47) for the fluid one whose density is every where the same. But the pressure on E is equal to the weight of the incumbent column of particles. If therefore $DE = x$, and $s =$ the specific gravity of the fluid at E, the weight of $DE = \int s dx$, hence

the pressure on Pp is $Pp . \int s dx$,

and $\therefore$ the pressure on BP is $\int Pp \int s dx$.

(1) To determine the pressure on the horizontal base of a vessel, supposing the density to vary as the depth.

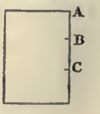
Let the whole depth $= a$, and the density at the base $= s'$,

$$\text{then } s = \frac{s'x}{a}, \text{ and } \int s dx = \frac{s'x^2}{2a} + C; \text{ but } C = 0;$$

$$\text{hence for the whole depth } \int s dx = \frac{s'a^2}{2a} = \tfrac{1}{2} s' a;$$

$\therefore$ the pressure on the base $=$ base $\times \tfrac{1}{2} s' a$.

(2) To find the pressure on any portion BC of a vertical side of a rectangular vessel, the density being supposed to vary as the depth.



$$\text{Here as before } s = \frac{s'x}{a}, \text{ and } \int s dx = \frac{s'x^2}{2a},$$

$$\text{whence the pressure} = \int Pp . \int s dx = \int \frac{s'x^2}{2a} . dx = \frac{s'x^3}{6a} + C.$$

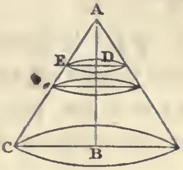
Now the pressure should $= 0$ when $x = AB$;

$$\therefore\ 0 = \frac{s'.AB^3}{6a} + C;$$

$$\therefore \text{ the whole pressure } = \frac{s'}{6a}.\{AC^3 - AB^3\}.$$

(3) A conical vessel is filled with fluid, whose density varies as the depth; the pressure on the base being equal to that on the sides, find the vertical angle.

Let $AB = a$, $BC = b$, $AD = x$, $DE = y$.



$$\text{The pressure on the base} = \frac{s'}{a}\int \pi b^2 x\,dx$$

$$= \frac{s'\pi b^2 x^2}{2a} = \frac{s'\pi b^2 a}{2}.$$

$$\text{Also } Ee = \frac{dx\sqrt{a^2+b^2}}{a};$$

$$\therefore \text{ the pressure on the side} = \int 2\pi y\,.\,Ee\int\frac{s'x\,dx}{a}$$

$$= \frac{s'\pi b}{a^2}.\frac{\sqrt{a^2+b^2}}{a}\int x^3\,dx$$

$$= \frac{s'\pi b\sqrt{a^2+b^2}}{a^3} \times \frac{x^4}{4} + C; \text{ but } C = 0;$$

and for the whole side, the pressure $= s'\pi b\sqrt{a^2+b^2} \times \frac{a}{4}$;

$$\therefore \frac{s'\pi b^2 a}{2} = \frac{s'\pi b a\sqrt{a^2+b^2}}{4},$$

$$\text{and } 2b = \sqrt{a^2+b^2};$$

$\therefore$ $BAC = 30^\circ$, and the vertical angle $= 60^\circ$.

Cor. The pressure on the base is equal to twice the weight of the fluid.

For $\pi y^2 =$ area of a section, and capacity $= \int \pi y^2\,dx$; and the density varying as x, $\int \pi y^2\,dx \times \frac{s'x}{a} =$ weight of fluid;

and $y^2 = \frac{b^2}{a^2} \cdot x^2$; $\therefore$ weight $= \frac{s' \pi b^2}{a^3} \cdot \int x^3 dx$

$$= \frac{s' \pi b^2}{a^3} \times \frac{x^4}{4}, \text{ since } C = 0;$$

$\therefore$ the whole weight $= \frac{s' \pi b^2 a}{4} =$ half the pressure on the base.

74. If gravity, instead of being constant, varied from one lamina to another, the pressure might be found. Thus, to find the pressure on a sphere surrounded by a fluid of uniform density which is acted on by a force tending to the centre of the sphere, and varying as $\frac{1}{D^2}$, the depth of the fluid being every where the same.

Suppose O to be the centre of the sphere, OA its radius, AD the depth of the fluid, and let $OA = a$, $OD = b$, the weight of a particle of fluid at the distance 1 from the centre $= g$;

$\therefore$ the weight of a particle at the distance $x = \frac{g}{x^2}$,

and the pressure on a small portion (m) of the surface of the sphere

$$= \int \frac{mg\,dx}{x^2} \text{ taken between } x = a \text{ and } x = b$$

$$= mg\left(\frac{1}{a} - \frac{1}{b}\right) = mg\,\frac{b-a}{ab};$$

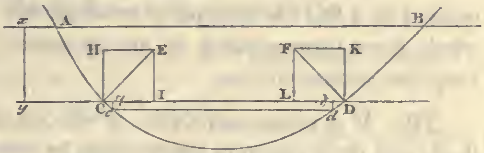
$\therefore$ the whole pressure on the sphere

$$= \frac{4\pi a^2}{m} \cdot mg \cdot \frac{b-a}{ab}$$

$$= 4\pi g a^2 \cdot \frac{b-a}{ab}.$$

75. Cor. If c be a mean proportional between a and b, the whole pressure $= 4\pi g a^2 \cdot \frac{b-a}{c^2}$, that is, it is equal to the weight of a cylinder of fluid (at the distance c) whose base is $= 4\pi a^2$ and height $= b - a$; or it is the weight of a cylinder whose base is the surface of the sphere, and height the depth of the fluid, at a distance OG, if we take $OA : OG :: OG : OD$.

76. The *vertical* pressure against the sides and bottom of any vessel is equal to the weight of the fluid in the vessel, if there be a vertical column over each portion, reaching to the surface.



Let AB be the surface of the fluid in the vessel; and parallel to it draw CD, cd indefinitely near to each other, so that each point in either of the intercepted portions Cc, Dd may be considered as at the same perpendicular depth xy. Draw CE, DF perpendicular to the surface at C and D, each $= xy$; then the perpendicular pressures on Cc and Dd may be represented by $s \times Cc \times CE$ and $s \times Dd \times DF$ (65). The pressure in the direction EC may be resolved into two, in the directions IC, HC; and these three forces will be proportional to the three sides of the triangle $Cc\gamma$ which are respectively perpendicular to the directions in which they act;

or the perpendicular : the horizontal pressure $:: Cc : c\gamma$;

$$\therefore \text{ the horizontal pressure } = s \times CE \times c\gamma.$$

In the same manner it may be shewn that the horizontal pressure at $D = s \times DF \times d\delta$ which is $= s \times CE \times c\gamma$;

$\therefore$ the pressures in the horizontal directions are equal and opposite.

And the vertical pressures are represented by

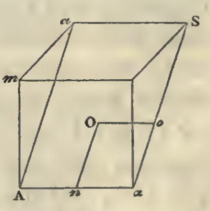
$$s \times EC \times C\gamma, \text{ and } s \times FD \times D\delta,$$
$$\text{or by } s \times xy \times C\gamma, \text{ and } s \times xy \times D\delta.$$

And the same may be proved for all other portions of the surface of the vessel; therefore the sum of the vertical pressures will be equal to the sum of the vertical superincumbent columns, or the whole vertical pressure will be equal to the weight of the fluid in the vessel.

77. Cor. The effect will be the same as if the whole had been solid, and consequently the same as if all the power were collected in the centre of gravity.

78. Def. The *centre of pressure* of a plane surface immersed in a fluid is that point to which if a force equal to the whole force were applied in an opposite direction, it would keep the surface at rest.

79. If a plane surface be produced to the surface of a fluid, and their common intersection be made the axis of suspension, the centres of pressure and percussion will coincide.



Let Sa represent a section of the plane, or plane produced, with the surface of the fluid, and let Sa be perpendicular to Sa in that plane; and let O be the centre of pressure. Draw Oo, Aa parallel to Sa; and On, Aa parallel to Sa. Then the force of the fluid against A is equal to the weight of a column of fluid whose base is A and altitude its perpendicular depth, and therefore it varies as $A \times Am \propto A \times Aa \times \sin\theta$, since $Am = Aa \times$ sine of inclination; and the effort of this to turn the plane round Oo

$$\propto A \times Aa \times On \times \sin\theta,$$
$$\propto A \times Aa \times (Aa - So) \,.\, \sin\theta,$$
$$\propto A \times Aa^2 \times \sin\theta - A \times Aa \times So \times \sin\theta;$$

therefore the effect of the forces on all the particles to turn the plane round Oo will be proportional to the sum of the $(A \times Aa^2) \,.\, \sin\theta$—$So \times$ sum of $(A \times Aa) \,.\, \sin\theta$, which $=0$ by the definition;

$$\therefore\ So = \frac{\text{sum of } (A \times Aa^2)}{\text{sum of } (A \times Aa)} = \frac{\text{sum of } (A \times Aa^2)}{\text{body} \times \text{dist. of centre of gravity}},$$

which is also the distance of the centre of percussion supposing Sa to be the axis of suspension, and the body to vibrate flatways.

Again, the effect of the force against A to turn the plane round $On \propto A \times Aa \times An$

$$\propto A \times Aa \times Aa - A \times Aa \times Oo;$$

therefore the effect of all the forces to turn the plane round On will be proportional to the sum of

$$A \times Aa \times Aa - A \times Aa \times Oo;$$

∴ the effect of all the forces to turn the plane round On will be proportional to

$$\text{the sum of } (A \times Aa \times Aa) - Oo \times \text{sum of } (A \times Aa) = 0;$$

$$\therefore\ Oo = \frac{\text{sum of } (A \times Aa \times Aa)}{\text{sum of } (A \times Aa)},$$

which is also the expression for the distance of the centre of percussion. Hence the centres of pressure and percussion coincide.

80. Let $Aa = x$, $Aa = y$, $A = dxdy$,

$$So = \frac{\iint x^2 dxdy}{\iint xdxdy} = \frac{\int x^2 ydx}{\int yxdx},$$

$$\text{and } Oo = \frac{\iint xydxdy}{\iint xdxdy} = \frac{\int y^2 xdx}{2\int yxdx}.$$

81. Cor. It is evident that the inclination of the surface pressed to the surface of the fluid has no effect in altering the centre of pressure, except when the two are parallel, in which case the centre of pressure will manifestly be the centre of gravity. And indeed in the expression

$$\sin\theta \times [\text{sum of the } (A \times Aa^2) - So \times \text{sum of the } (A \times Sa)] = 0,$$

$\sin\theta$ itself is $= 0$; ∴ we cannot assume the sum of the

$$(A \times Aa^2) = So \times \text{sum of the } (A \times Sa).$$

(1) Find the centre of pressure of a vertical line whose length is l.

Let the distance of the upper extremity of the line from the surface of the fluid $= a$, and of the lower $= b$; therefore $a + l = b$, and the distance of the centre of pressure from the surface of the fluid $= \dfrac{\int x^2 y dx}{\int x y dx} = \dfrac{2.(x^3 - a^3)}{3.(x^2 - a^2)}$, since y is constant; and when $x = b$, the distance $= \dfrac{2.(b^3 - a^3)}{3.(b^2 - a^2)}$.

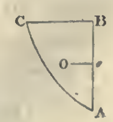
Cor. 1. If one extremity of the line coincide with the surface of the fluid, $a = 0$, and $b = l$; $\therefore$ the distance $= \frac{2}{3} l$.

Cor. 2. If on the vertical line, a rectangle be described, the distance of its centre of pressure from the surface $= \dfrac{2(b^3 - a^3)}{3.(b^2 - a^2)}$, which will evidently be measured in the vertical line which bisects the rectangle.

Cor. 3. Let $s =$ the specific gravity of the fluid and $c =$ the horizontal base of the rectangle; and suppose the upper side of the rectangle to coincide with the surface of the fluid, the distance of the centre of pressure $= \frac{2}{3} l$, and the tendency of the rectangle to turn round the base $= \frac{1}{2} scl^2 \times \frac{1}{3} l = \frac{1}{6} scl^3$. But the tendency to turn round one of the vertical sides $= \frac{1}{2} scl^2 \times \frac{1}{2} c = \frac{1}{4} sc^2 l^2$;

$\therefore$ the first of these efforts : the second :: $2l : 3c$, which when the figure is a square, becomes as $2 : 3$.

(2) To find the centre of pressure of a semi-parabola, the extreme ordinate coinciding with the surface of the fluid.



Let $AB = a$, $BC = b$, and suppose O the centre of pressure, and draw Oo parallel to BC; then if x and y represent the abscissa and ordinate of the curve $y = \dfrac{bx^{\frac{1}{2}}}{a^{\frac{1}{2}}}$;

$$\text{and } Bo = \frac{\int (a-x)^2 \cdot \frac{b x^{\frac{1}{2}}}{a^{\frac{1}{2}}} \cdot dx}{\int (a-x) \cdot \frac{b x^{\frac{1}{2}}}{a^{\frac{1}{2}}} \cdot dx} = \frac{\int (a^2 x^{\frac{1}{2}} dx - 2 a x^{\frac{3}{2}} dx + x^{\frac{5}{2}} dx)}{\int (a x^{\frac{1}{2}} dx - x^{\frac{3}{2}} dx)}$$

$$= \frac{\frac{2}{3} a^2 x^{\frac{3}{2}} - \frac{4}{5} \cdot a x^{\frac{5}{2}} + \frac{2}{7} x^{\frac{7}{2}}}{\frac{2}{3} a x^{\frac{3}{2}} - \frac{2}{5} x^{\frac{5}{2}}} = \frac{\frac{2}{3} a^2 - \frac{4}{5} a x + \frac{2}{7} x^2}{\frac{2}{3} a - \frac{2}{5} x}, \text{ } C \text{ being} = 0,$$

$$\text{and when } x = a, \; Bo = \frac{4a}{7}.$$

$$\text{Also } Oo = \frac{\int \frac{b^2}{a} \cdot x \cdot (a-x)\, dx}{2 \int \frac{b}{a^{\frac{1}{2}}} \cdot (a-x) \cdot x^{\frac{1}{2}} dx} = \frac{b}{2 a^{\frac{1}{2}}} \cdot \frac{\int (a x\, dx - x^2 dx)}{\int (a x^{\frac{1}{2}} dx - x^{\frac{3}{2}} dx)}$$

$$= \frac{b}{2 a^{\frac{1}{2}}} \cdot \frac{\frac{1}{2} a x^2 - \frac{1}{3} x^3}{\frac{2}{3} a x^{\frac{3}{2}} - \frac{2}{5} x^{\frac{5}{2}}} = \frac{b}{2 a^{\frac{1}{2}}} \cdot \frac{\frac{1}{2} a x^{\frac{1}{2}} - \frac{1}{3} x^{\frac{3}{2}}}{\frac{2}{3} a - \frac{2}{5} x},$$

$$\text{and when } x = a, \; Oo = \frac{5b}{16}.$$

(3) A parallelopiped with its sides vertical has one side loose which revolves round a hinge at the bottom, and is kept in its position by a given pressure applied at a given point; how high may the vessel be filled with fluid before the side will be forced open.

Let p be the given pressure, a the breadth of the side, b the distance from the bottom at which it acts, v the required altitude; then considering the density of the fluid as represented by s, the pressure of it against the side

$$= s \cdot a v \cdot \frac{v}{2} = \frac{s a v^2}{2},$$

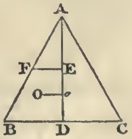
and this takes place at the centre of pressure, the distance of which from the surface (if x is any variable distance) is

$$\frac{\int a x^2 dx}{\int a x dx} = \frac{\frac{x^3}{3}+C}{\frac{x^2}{2}+C'} = \frac{2}{3}v, \text{ from } x=0 \text{ to } x=v.$$

Hence, by the Problem, $pb = \frac{sav^2}{2} \cdot \frac{v}{3} = \frac{sav^3}{6}$;

$$\therefore\ v = \left(\frac{6pb}{sa}\right)^{\frac{1}{3}}.$$

4. *ABC* is the side of a tetrahedron which is filled with fluid, and *AD* a perpendicular from the vertex on *BC*. If *ABD* be loose, find the magnitude, point of application, and inclination to the horizon, of a single force which will keep it at rest; 1. when the vessel rests on its base; 2. when it rests on its vertex; the base in each case being horizontal.



In the first case, let O be the center of pressure, then $Ao = \frac{\int x^2 y dx}{\int x y dx}$. If $AE = x$, $EF = y$, $AD = a$, $DB = b$, $AB = c$; by similar triangles $y = \frac{b}{a} \cdot x$;

$$\therefore \int x^2 y dx = \frac{b}{a} \int x^3 dx = \tfrac{1}{4} \cdot \frac{b}{a} \cdot x^4 + C, \text{ and } C = 0;$$

$$\therefore \int x^2 y dx = \tfrac{1}{4} \cdot b a^3, \text{ when } x = a.$$

And $\int x y dx = \frac{b}{a} \cdot \int x^2 dx = \frac{1}{3} \cdot \frac{b}{a} \cdot x^3 + C = \frac{1}{3} b a^2$, when $x = a$;

$$\therefore\ Ao = \frac{\frac{1}{4} b a^3}{\frac{1}{3} b a^2} = \tfrac{3}{4} a = \tfrac{3}{4} \cdot c \cdot \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{8} \cdot c.$$

Again, $Oo = \dfrac{\frac{1}{2}\int x y^2 dx}{\int x y dx}$,

$$\text{and} \int x y^2 dx = \frac{b^2}{a^2}\int x^3 dx = \frac{b^2}{a^2}\cdot\frac{x^4}{4} + C$$

$$= \frac{b^2}{a^2}\cdot\frac{a^4}{4} = \tfrac{1}{4} b^2 a^2 \text{ when } x = a;$$

$$\therefore\ Oo = \frac{\frac{1}{2}\cdot\frac{b^2 a^2}{4}}{\frac{1}{3} b a^2} = \frac{3}{8} b = \frac{3}{8}\cdot\frac{c}{2} = \frac{3}{16} c.$$

The magnitude of the force to be applied at $O =$ the weight of a cylinder of fluid whose base $= ABD$ and altitude the perpendicular depth of the centre of gravity of ABD (64), $= \dfrac{ab}{2} \times \frac{2}{3} AD \times$ sine of inclination of a side of the tetahedron to the base, $= \dfrac{ab}{2}\cdot\dfrac{2}{3}a\cdot\dfrac{2\sqrt{2}}{3}$

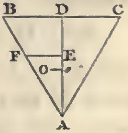
$$= \frac{2\sqrt{2}}{9}\cdot a^2 b = \frac{2\sqrt{2}}{9}\cdot\frac{3}{4}c^2\cdot\frac{c}{2} = \frac{c^3}{6\sqrt{2}},$$

the specific gravity of the fluid being represented by 1.

The direction of the force, being perpendicular to the side whose inclination to the horizon is an angle whose sine $= \dfrac{2\sqrt{2}}{3}$, will be inclined to the horizon at an angle whose cosine $= \dfrac{2\sqrt{2}}{3}$, or whose sine $= \dfrac{1}{3}$.

In the second case

$$Do = \frac{\int DE^2 . y dx}{\int DE . y dx} = \frac{\frac{b}{a}\cdot\int (a-x)^2 . x dx}{\frac{b}{a}\cdot\int (a-x) . x dx}$$



$$= \frac{\int (a-x)^2 . x dx}{\int (a-x) . x dx} = \frac{\frac{1}{12}a^3}{\frac{1}{6}a^2} = \tfrac{1}{2} a = \frac{c\sqrt{3}}{4},$$

And $$Oo = \frac{\frac{1}{2}\int DE \,.\, y^2 dx}{\int DE \,.\, y dx} = \frac{\frac{1}{2}\frac{b^2}{a^2}\int(a-x)\,.\,x^2 dx}{\frac{b}{a}\,.\int(a-x)\,.\,x dx}$$

$$= \frac{b}{2a} \,.\, \frac{\frac{1}{12}a^4}{\frac{1}{6}a^3} = \frac{b}{4} = \frac{c}{8}.$$

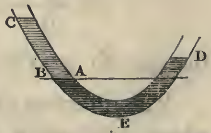
The magnitude of the force is

$$\frac{ab}{2} \,.\, \frac{1}{3}a \,.\, \frac{2\sqrt{2}}{3} = \frac{\sqrt{2}}{9} \,.\, a^2 b = \frac{\sqrt{2}}{9} \,.\, \frac{3}{4}c^2 \,.\, \frac{c}{2} = \frac{c^3}{12\sqrt{2}} =$$

$\frac{1}{2}$ the former. The direction is the same as before.

Cor. Since in the first case $Ao = \frac{3}{4}a$, and in the second $Do = \frac{1}{2}a$, it appears that the distance of the centre of pressure of the triangle from the intersection of the plane and the surface of the fluid is in one case $\frac{3}{4}$ths of the axis, and in the other $\frac{1}{2}$ of the axis.

82. The perpendicular altitudes of fluids communicating through a bent tube vary inversely as their specific gravities.



Let two fluids whose specific gravities are S and s communicate through a tube CED, and suppose a plane to pass through the tube contiguous to their common surface AB, and let the perpendicular depths of its centre of gravity below the surfaces C and D be G and g. Then the pressures of the fluids in CB and DEB against this plane (65) are to each other $:: G \times S : g \times s$. But the fluids being at rest, these pressures are equal; $\therefore\ G \times S = g \times s$,

$$\text{or } G : g :: s : S,$$

that is, the perpendicular altitudes are inversely as the specific gravities.

Ex. If the specific gravities of the fluids are as $14 : 1$, $G : g :: 1 : 14$, and therefore to balance a column of the latter 35 feet high, a column of the former of the altitude of 30 inches will be required.

83. Cor. 1. If the fluids have the same specific gravity,

they will stand at the same perpendicular altitude in the opposite branches of the tube.

84. Cor. 2. The proposition is true, whatever be the shape of the branches of the tube. The diameter must exceed $\frac{1}{15}$th of an inch.

85. Cor. 3. Hence appears the reason why the surface of small pools near rivers are always on a level with the surfaces of the rivers, when there is any subterraneous communication between the rivers and the pools. These may be considered as communicating vessels.

86. Cor. 4. Hence also if water be conveyed by pipes from a distance, as in the case of public fountains, when the point to which it is conveyed, is of the same altitude as that from which it proceeds, the surface will be at rest: if lower, it will, by endeavouring to rise to the same altitude, cause a stream to flow.

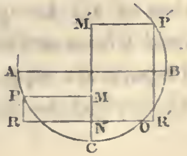
87. Cor. 5. Hence though water may be conveyed to all intermediate distances, it cannot be conveyed to a greater height than the surface of the stagnant fluid.

88. Cor. 6. Since $G \times S = g \times s$, if S' and s' be the specific gravities of two other fluids which are placed above C and D, and the equilibrium still remains, G' and g' being their perpendicular altitudes above C and D,

$$G \times S + G' \times S' = g \times s + g' \times s'.$$

And, in general, if there be any number of fluids in equilibrio in the opposite branches of a tube, the sums of the products of their heights and specific gravities in each branch will be equal.

Prob. Supposing equal lengths of two fluids, whose specific gravities are as $m : 1$, to be poured into a circular tube; determine their position when at rest.



Let the fluids occupy the spaces CA, CB at first, each being $= a$; and let the heavier descend to P, and the lighter ascend to P'; and let O be the point where the surfaces meet. Draw PM, $P'M'$, ON

perpendicular to CMM', and PR, $P'R'$ perpendicular to RR',

then $M'N : MN :: P'R' : PR :: m : 1$; $\therefore M'N = m.MN$.

Let $CP = x$; $\therefore CO = a - x$, and $CP' = a - x + a = 2a - x$,

$$\text{whence } CN = \text{versin}\,(a - x), \quad CM' = \text{versin}\,(2a - x),$$

$$CM = \text{versin}\,x,$$

$$\text{and } NM' = CM' - CN = \text{versin}\,(2a - x) - \text{versin}\,(a - x),$$

$$NM = CM - CN = \text{versin}\,x - \text{versin}\,(a - x).$$

$$\text{Hence versin}\,(2a - x) - \text{versin}\,(a - x)$$

$$= m\,.\,\text{versin}\,x - m\,.\,\text{versin}\,(a - x),$$

$$\text{or} \quad 1 - \cos(2a - x) - 1 + \cos(a - x)$$

$$= -m\,.\cos x + m\,.\cos(a - x);$$

$$\therefore \cos(2a - x) + (m - 1)\,.\cos(a - x) - m\,.\cos x = 0,$$

$$\text{and} \quad \cos 2a\,.\cos x + \sin 2a\,.\sin x + (m - 1)\,.\cos a \times \cos x$$

$$+ (m - 1)\,.\sin a\,.\sin x - m\,.\cos x = 0;$$

$$\therefore [\cos 2a + (m - 1)\,.\cos a - m]\,.\cos x$$

$$+ [\sin 2a + (m - 1)\,.\sin a]\,.\sin x = 0;$$

$$\therefore \tan x = -\frac{\cos 2a + (m - 1)\,.\cos a - m}{\sin 2a + (m - 1)\,.\sin a},$$

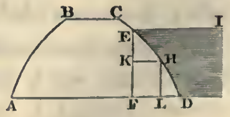
whence CP may be determined by the tables.

Cor. Let $a = \frac{\pi}{2}$, then $\cos 2a = -1$, $\cos a = 0$, $\sin 2a = 0$, $\sin a = 1$;

$$\text{and } \tan x = -\frac{-1 - m}{m - 1} = \frac{m + 1}{m - 1}.$$

89. Def. A *dyke* is a mound or obstacle opposed to the effort made by a fluid to spread itself.

90. Let $ABCD$ be a vertical section of a dyke opposed to the stagnant fluid whose surface is EI and depth EF. Its parts are supposed to be so connected as to yield to the pressure of the fluid, either by turning



altogether round the point A, or by sliding along the horizontal base DA.

91. Supposing the dyke to yield, by turning round A, to determine when there will be an equilibrium.

The effort to overthrow the dyke arises from the force which the fluid exerts horizontally; and the stability is caused by the vertical pressure of the fluid on the curve ED, and the weight of the dyke. When therefore there is an equilibrium, the former of these forces must be equivalent to the two latter. Let EH be any portion of the curve ED, draw HL perpendicular, and HK parallel to AD; and let $EH = z$, $HK = x$, $EK = y$, $EF = a$, $AD = b$, $FD = c$.

If the increment of EH be represented by dz, the perpendicular pressure on its surface $= sydz$ (64) (1 being taken $=$ the horizontal breadth, and $s =$ the specific gravity of the fluid); ∴ the horizontal force on EH

$$= \int sydz \times \frac{dy}{dz} = s\int ydy = \tfrac{1}{2}sy^2,$$

and the perpendicular distance from E at which the force must be applied, equivalent to the sum of these pressures, $= \dfrac{\int y^2 dy}{\int y dy} = \tfrac{2}{3}y$; C being $= 0$, since the numerator and denominator vanish with y. Let now $y = a$, and the horizontal force to turn the body round $A = \frac{1}{2}sa^2 \times \frac{1}{3}a = \frac{1}{6}sa^3$.

The vertical pressure on the increment of the curve at $H = sydx$, and the force of this to turn the body round A

$$\begin{aligned} &= sydx \times AL \\ &= sydx\,.\,(AD - FD + FL) \\ &= sydx\,.\,(b - c + x), \end{aligned}$$

and the sum of these forces $= s\,.\int(b - c + x)\,.\,ydx$, from E to D.

If s' be the specific gravity of the dyke, and Q the product of the area $ABCD$ multiplied by the distance of A from the vertical passing through the centre of gravity of the area; the effort of the weight of the dyke to turn it round $A = s'Q$.

Hence therefore in the case of equilibrium

$$\tfrac{1}{6}\, s a^3 = s' Q + s\int (b - c + x)\,.\,y\,dx.$$

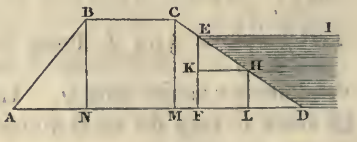
92. Supposing the dyke to yield by sliding along its horizontal base: to determine when there will be an equilibrium; neglecting the vertical pressure of the fluid.

The base being horizontal, the mass which it sustains is supported against the horizontal force of the fluid, only by its adhesion to the base, and the resistance arising from friction. Suppose these resistances $=n$ times the weight of the dyke, (n being to be determined by experiment); and let $P=$ the area of the section $ABCD$; $s'P$ will be its weight, and $ns'P$ the resistance to the horizontal force of the fluid, which $=\frac{1}{2}sa^2$,

$$\therefore\ P = \frac{s}{s'} \times \frac{a^2}{2n}.$$

If the vertical pressure of the fluid be considered, it must be added to the weight of the dyke. If it be neglected, the computed dimensions of the dyke are more advantageous for resistance.

93. If the sides of the dyke be rectilinear, and AD, BC horizontal; to determine the equation of equilibrium of Art. 91.



Let fall the perpendiculars CM, BN. Let $CM = h$, $MD = e$, $AN = e'$.

By similar triangles EKH, EFD, $x = \frac{c}{a}\,.\,y$;

$$\therefore \int (b - c + x)\,.\,y\,dx = \int \left(b - c + \frac{c}{a}\,.\,y\right).\,\frac{c}{a}\,.\,y\,dy$$

$$= \frac{bc}{2a}\,.\,y^2 - \frac{c^2}{2a}\,.\,y^2 + \frac{c^2}{3a^2}\,.\,y^3 + C;$$

but $C = 0$, since the whole vanishes when $y = 0$.

Let $y = a$; it becomes $= \frac{1}{2}bca - \frac{1}{2}c^2a + \frac{1}{3}c^2a = \frac{1}{2}bca - \frac{1}{6}c^2a$.

And $Q=(b-e-e').h.\{e'+\frac{1}{2}(b-e-e')\}+\frac{he}{2}.(b-\frac{2}{3}e)+$

$$\frac{he'}{2}\times\tfrac{2}{3}e'=\tfrac{1}{2}hb^2-\tfrac{1}{2}hbe+\tfrac{1}{6}h.\{e^2-e'^2\}.$$

Let these values be substituted in the general equation,
$\therefore \frac{1}{6}sa^3=\frac{1}{2}sbca-\frac{1}{6}sc^2a+\frac{1}{2}s'hb^2-\frac{1}{2}s'hbe+\frac{1}{6}s'h.(e^2-e'^2),$

whence $b^2+\left\{\frac{sc}{s'h}.a-e\right\}.b-\frac{s.(a^3+c^2a)}{3s'h}+\frac{1}{3}(e^2-e'^2)=0,$

an equation, which includes all the cases of rectilinear sloping banks.

94. Cor. If the slopes be $=0$, or the dykes vertical, $e=0$, $e'=0$ and $c=0$;

$$\therefore\ b^2-\frac{sa^3}{3s'h}=0,$$

$$\text{and}\quad b=\sqrt{\frac{sa^3}{3s'h}}.$$

95. If the sides be rectilinear, and AD, BC horizontal, to determine the equation of equilibrium of Art. 92.

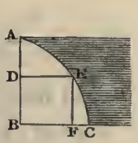
The area $ABCD=hb-\frac{1}{2}h.(e+e')$;

$$\therefore\ hb-\tfrac{1}{2}h.(e+e')=\frac{s}{s'}\times\frac{a^2}{2n},$$

$$\text{and}\quad b=\frac{s}{s'}\times\frac{a^2}{2hn}+\tfrac{1}{2}.(e+e').$$

96. The preceding equations have been investigated on the supposition of a perfect connexion of all the parts of the dyke; they are therefore only applicable to such as are constructed of masonry. In those which are composed of earth, with slopes fortified by stone or by other means, the constituent parts have not the same connexion that those of masonry have; and therefore though satisfying the preceding formulæ, when taken for the whole height, they will nevertheless not

resist equally in the several parts of that height, but divide into horizontal sections.



97. If ABC be the section of a dyke whose summit is on a level with the surface of the fluid. To determine the nature of the curve AEC, so that each portion AED may remain in equilibrio on its base ED; not separating from the lower part $DECB$, either by turning round D, or sliding along ED.

Let $AD = x$, $DE = y$. The horizontal force of the fluid at $D = \frac{1}{6} . s x^3$ (91); and the momentum of the section ADE with reference to the point $D = s' \int y \, dx \times$ the horizontal distance of its centre of gravity from D

$$= s' \int y \, dx . \frac{\frac{1}{2} \int y^2 dx}{\int y \, dx} = \frac{1}{2} s' \int y^2 dx ;$$

$$\text{hence } \frac{1}{6} . s x^3 = \frac{1}{2} s' \int y^2 dx,$$

$$\text{or } \frac{1}{2} s x^2 dx = \frac{1}{2} s' y^2 dx,$$

$$\text{and } \sqrt{s} . x = \sqrt{s'} . y ;$$

the figure of the slope therefore is a rectilinear triangle whose base BC : height AB :: $\sqrt{s} : \sqrt{s'}$.

The conditions necessary for preventing AED from sliding on its base ED, will be determined from the equation

$$\frac{1}{2} s x^2 = n s' \int y \, dx ;$$

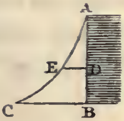
$$\therefore \; s x \, dx = n s' y \, dx,$$

$$\text{and } s x = n s' y,$$

or $x : y :: n s' : s$, where n is determined by experiment.

98. If ACB be the section of a wall supporting a fluid against its vertical side AC : to determine the nature of the curve AEB bounding the other side, so that its strength may be in every part proportional to the pressure it sustains.

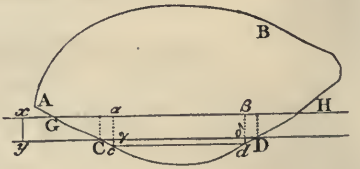
It may be shewn as in Art. 91, that the horizontal force to turn the body round $D = \frac{1}{6}s \, . \, DA^3$, and therefore this force $\propto DA^3$. Now by Mechanics the strength of the wall $\propto DE^2$, whence $DA^3 \propto DE^2$, or the curve must be a semi-cubical parabola whose vertex is A, and convex towards AC.



Sect. III.

99. Def. The *Plane of Floatation* is the horizontal surface of the fluid in which the body floats.

100. A body floating on the surface of a fluid is pressed *upwards* by a force equal to the weight of a quantity of fluid of the same magnitude with the part of the body immersed.



Let $ABDC$ be a body floating on a fluid, in which the part immersed is $GCDH$, GH being the plane of floatation. Draw CD, cd parallel to GH, and indefinitely near to each other, so that each portion of Cc and Dd may be considered as at the same perpendicular depth. Then making the same construction, and proceeding as in Art. 76, it may be proved that the horizontal pressures are equal and in opposite directions, and that the vertical pressures upwards on Cc, Dd may be represented by $xy \times C\gamma$, $xy \times D\delta$. And the same being also true for every other portion of the immersed surface, the pressure upwards will be equal to the sum of the weights of the columns $C\alpha$, $D\beta$, &c. of fluid, that is, to the weight of a quantity of fluid equal in magnitude to the part immersed.

101. Cor. 1. If the solid float in equilibrio, the pressures downwards and upwards are equal; that is, the weight of the floating body is equal to the weight of a quantity of fluid of the same magnitude with the part of the body immersed.

Prob. The nth part of a hollow paraboloid with its vertex downwards is filled with a fluid of known specific gravity, and a sphere of given size and substance is placed in it. Find how high the fluid will rise.

Let $r =$ the radius of the sphere,
s and $s' =$ the specific gravities of the sphere and fluid respectively,
$a =$ the length of the axis of the paraboloid,
and $l =$ its latus rectum;

$\therefore \frac{4}{3}\pi r^3 s =$ weight of the sphere $=$ magnitude immersed $\times\, s'$,

whence the magnitude immersed $= \dfrac{4\pi r^3 s}{3s'}$.

Also $\dfrac{\pi l a^2}{2n} =$ the part of the paraboloid occupied by the fluid at first;

$\therefore$ if $x =$ the depth of the fluid afterwards,

$$\frac{\pi l x^2}{2} = \frac{4\pi r^3 s}{3s'} + \frac{\pi l a^2}{2n},$$

$$\text{and } x = \sqrt{\frac{8r^3 sn + 3la^2 s'}{3nls'}}.$$

102. **Cor. 2.** In a similar manner it may be shewn that if a solid be *immersed* in a fluid, the pressure upwards against the base is equal to the weight of a quantity of fluid of the same magnitude with the solid, together with the weight of the superincumbent fluid.

103. **Cor. 3.** Hence the difference between the pressures downwards and upwards is equal to the difference between the weights of the solid and of an equal bulk of fluid.

104. **Cor. 4.** Hence a solid immersed in a fluid of the same specific gravity with itself, will remain at rest in all positions.

For the vertical pressures are equal, since the magnitudes and specific gravities are the same; therefore the body cannot move vertically; nor can it move horizontally, since the horizontal pressures are equal and in opposite directions

(76); and as an oblique motion is compounded of a vertical and a horizontal one, and these are equal and opposite; therefore the body cannot have any oblique motion.

105. The ascent of a solid in a fluid specifically heavier than itself, is owing to the pressure upwards of the fluid.

For if a solid ascend, it must be acted upon by some force tending upwards, otherwise it would descend by its own gravity; but there is no force tending upwards except the pressure of the fluid in that direction; it is therefore that pressure by which it is compelled to ascend.

106. A solid immersed in a fluid of less or greater specific gravity than itself, will descend or ascend with a force equal to the difference between the weights of the solid and an equal bulk of fluid.

For (103) the difference between the pressures downwards and upwards is equal to the difference between the weights of the solid and of an equal bulk of fluid; that is, (if $M =$ the magnitude of the solid, and S and s the specific gravities of the solid and fluid) $= M.(S \sim s)$. If then the solid be specifically lighter than the fluid, the force upwards $= M.(s - S)$; and if heavier, the force downwards $= M.(S - s)$.

This also will appear from the following experiment:

If a wooden plate be closely fitted to the bottom of a vessel, and then mercury be poured into the vessel, it will remain unmoved: but if the plate be so disengaged as to admit the mercury between itself and the bottom of the vessel, it will instantly ascend*. Hence, therefore, when the pressure of the fluid upwards is removed, the body does not ascend though specifically lighter than the fluid; (for in this case it is not pressed upwards by any force, whilst it is pressed downwards not only by its own gravity, but also by the weight of the superincumbent column of fluid), whereas when the pressure upwards is not removed, it does ascend. We may infer therefore that the ascent is owing to the pressure upwards of the fluid.

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* Mercury is fitter for this experiment than water, since it does not adhere to wet wood as water does; and therefore cannot so easily insinuate itself between the plate and the bottom on which it rests.

107. Cor. The accelerating force will be the difference of the weight of the solid and equal bulk of fluid, divided by the mass of the body. And whilst the body is entirely immersed, this force will be constant, whence the motion will be uniformly accelerated; unless disturbed by the resistance of the medium.

If the body is specifically heavier than the fluid, it will descend to the bottom, and press it with the excess of its weight above that of an equal bulk of fluid. If lighter it will ascend and float, and rise till the weight of the body is equal to that of a quantity of fluid of the same magnitude as the part immersed.

108. The pressure upwards may be made so great as to sustain a body whose specific gravity exceeds that of the fluid in any assignable ratio*.

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* This principle is employed to raise heavy bodies from the bottom of rivers, or ships which have sunk. The weight to be raised is fastened to a large vessel previously loaded so as to sink deep; and upon the removal of the lading, the pressure upwards of the water causes the vessel, and with it the attached weight, to ascend. It has also been employed to carry ships over shoals, by fastening to their bottoms large chests filled with water, which being afterwards pumped out rendered them sufficiently buoyant, and enabled them to pass over the shallows.

It has also given rise to the construction of the Camel: which consists of two half ships made so that they can be applied below water on each side of a large vessel. When they are to be used, they are filled with water sufficient to sink them to the proper depth; and the vessel being secured between them, the water is pumped out: they therefore ascend, and with them the vessel to which they are attached.

For the pressure upwards (100) is equal to the weight of a column of fluid having the same base as the solid, and an altitude equal to the depth below the surface of the fluid: which, as the base and specific gravity are given, will increase with the increase of the depth, and may therefore become greater than any assignable quantity. Now the pressure of the fluid downwards being removed, the only remaining pressure downwards is the weight of the solid, which is the same at all depths; therefore the pressure on the base of the solid may be increased till it sustains the weight, however great, of the solid.

This will explain the common experiment of making lead swim, in consequence of its being fitted to the bottom of a glass tube.

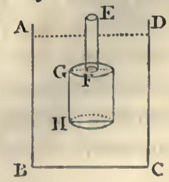
109. Having given the specific gravities of a cylindrical solid of known thickness; to determine the depth at which the solid will be just supported.

Suppose the solid just supported: then (102) the pressure upwards on the base of the solid is the weight of a cylinder of fluid whose base is (B) the base of the solid, and altitude the perpendicular depth (D) of that base, $= B \times D \times S$; and the weight of the solid $= B \times H \times S'$; S and S' being the specific gravities of the fluid and solid, and H the given thickness; but these weights are equal; $\therefore\ D \times S = H \times S'$,

$$\text{or } D = \frac{H \times S'}{S}.$$

Prob. In a vessel of fluid, a hollow cylinder EF is placed, to the bottom of which a cylindrical body of greater specific gravity than the fluid is so closely attached that no fluid can enter. To determine the depth at which the body will rest.

Let a and b be the areas of the upper surface of the body and cylinder, $c =$ thickness of the body $= GH$, $x =$ the depth required, and s and s' the specific gravities of the body and the fluid.



The part of the upper surface which is pressed downwards is $a - b$, and its depth being x, the pressure downwards on it $= (a-b) \,.\, s'x$, and the weight of the body $= sac$;

$$\therefore \text{ the whole force downwards} = (a-b) \,.\, s'x + sac,$$

$$\text{and the force upwards} = s'a \,.\, (x+c);$$

$$\text{hence } s'a \,.\, (x+c) = (a-b) \,.\, s'x + sac;$$

$$\therefore s'ac = sac - s'bx,$$

$$\text{and } x = \frac{(s-s') \,.\, ac}{s'b}.$$

Cor. If r and r' be the radii of the surface and the cylinder,

$$x = \frac{(s-s') \,.\, cr^2}{s'r'^2}.$$

110. When a body is immersed in a fluid, the weight lost is to the whole weight as the specific gravity of the fluid is to that of the solid.

When a body is immersed in a fluid, the force with which it descends will manifestly be equal to the difference between the weight of the solid (W) and the weight lost by the action of the fluid (W'). But it has been shewn (106) that the force with which it descends is the difference between the weights of the solid and of an equal bulk of fluid $= W - w$; whence

$$W - W' = W - w,$$

$\therefore W' = w = Ms'$, if s' be the specific gravity of the fluid, and M the magnitude of the solid. Hence

$$W' : W :: Ms' : Ms :: s' : s.$$

111. The part of the weight thus lost is not destroyed: it is only sustained by a weight acting in a contrary direction. Hence it is evident, why the weight of a bucket of water is not perceived while it is in the water. It is not because fluids do not gravitate while they are in fluids similar to themselves: but because there is a pressure in a contrary direction which is precisely equal to their gravity.

112. Cor. 1. If the specific gravities of the fluid and body are as $1 : n$; and $W =$ the weight of the body, the weight lost $= \frac{W}{n}$; therefore the weight of the body in the fluid

$$= W - \frac{W}{n} = W . \frac{n-1}{n}.$$

113. Cor. 2. The weights which bodies lose in different fluids are as the specific gravities of those fluids.

Prob. Having given the weight of a body in air and water; to determine its real weight.

Let $x =$ its real weight; w and w' its weights respectively in air and water, then $x - w$ and $x - w'$ are the weights lost, and if s and s' be the specific gravities of air and water,

$$x - w : x - w' :: s : s';$$

$$\therefore s'x - s'w = sx - sw',$$

$$\text{or } (s' - s) . x = s'w - sw',$$

$$\therefore x = \frac{s'w - sw'}{s' - s}.$$

If the body be a globe whose diameter is required, as also its specific gravity; let $y =$ its specific gravity,

$$\therefore w : x :: y - s : y,$$

$$\text{and } x : w' :: y : y - s';$$

$$\therefore w : w' :: y - s : y - s',$$

$$\text{whence } y = \frac{s'w - sw'}{w - w'},$$

and if $\delta =$ the diameter, the magnitude $= \frac{\pi \delta^3}{6}$;

$$\therefore \frac{\pi \delta^3}{6} \times \frac{s'w - sw'}{w - w'} = x = \frac{s'w - sw'}{s' - s};$$

$$\therefore \delta = \sqrt[3]{\frac{6}{\pi} \times \frac{w - w'}{s' - s}}.$$

114. Cor. 3. Bodies of equal weights, but different magnitudes, lose in the same fluid weights which are reciprocally proportional to their specific gravities, or which are directly as their magnitudes.

115. Cor. 4. When a body is weighed in air, in order to obtain its absolute weight there must be added to it the weight of an equal bulk of air. For (110), $W'=w$.

116. Cor. 5. If two bodies, of the same weight in air, be put into a denser fluid, the smaller body will preponderate: since it loses a less weight than the other (113).

117. Cor. 6. If two bodies weigh equally in any fluid, and then be brought into a rarer medium, the greater will preponderate.

For the weight lost in the denser fluid $= MS$, in the rarer $= Ms$; $\therefore$ the increase of weight from a body's being brought into a rarer fluid $= M\,.\,(S-s)$, which $\propto M$; hence the greater body will preponderate.

118. Cor. 7. Bodies which sustain equal losses of weight are of the same magnitude.

For the losses of weights are as the weights of the quantities of the fluid displaced, which are as the magnitudes of the bodies that displace them. And therefore if the losses of weights be equal, the magnitudes must also.

119. It may here be observed, that if a body suspended in the air be acted upon by gravity, its moving force in this fluid is equal to the excess of its weight above that of the air which it displaced; so that it is acted upon by a constant accelerating force a little less than that of gravity *in vacuo*. To compare these two forces, let g represent the first, and G the second; M the magnitude of the body, S and s the densities of the body and air.

Then $MD =$ the mass, and $GMS - GMs =$ the moving force;

$$\therefore \text{ the accelerating force } g = G\,.\left(1 - \frac{s}{S}\right).$$

Hence the air retards the motions of heavy bodies on two accounts, 1. because it acts as a resisting medium, which produces a retarding force depending upon the velocity of the moving body, 2. it diminishes the constant accelerating force. The first cause has no influence on the time of a whole oscillation; but this is not the case with the second. The time varies inversely as the square root of the force, and therefore depends upon the density of the air in which the oscillations are performed, and will increase with that density. The value of g therefore determined by observations of the pendulum in the air, does not give the force of gravity *in vacuo*. But by means of the preceding formula we may calculate the value of G from that of g; knowing the ratio $\frac{s}{S}$.

Prob. If a piece of copper is exactly counterpoised in air by a brass weight, and both the copper and brass are immersed in a fluid, the copper will preponderate (116); to determine what weight must be added to restore the equilibrium.

Let $w =$ the weight of either solid; c and b the specific gravities of copper and brass, s the specific gravity of the fluid in which they are immersed.

The brass will lose by immersion a weight $= \frac{ws}{b}$,

and the copper a weight $= \frac{ws}{c}$;

the difference therefore of these will be a weight (w') by which the equilibrium is destroyed; or

$$w' = \frac{ws}{b} - \frac{ws}{c} = ws \cdot \frac{c-b}{cb},$$

the weight which must be added (not immersed) to restore the equilibrium.

1. If 120 grs. of brass are counterpoised in air by a piece of copper, and both are immersed in water,

$$w' = 120 \,.\, \frac{9-8}{9 \times 8} = \frac{120}{72} = 1\tfrac{2}{3} \text{ grs.}$$

the weight to be applied to the ascending arm of the balance to restore the equilibrium.

2. If a guinea weighing 129 grs. be counterpoised by a piece of brass, and both the gold and brass are immersed in water,

$$w' = 129 \times \frac{17.2-8}{8 \times 17.2} = 8\tfrac{1}{2} \text{ grs.} +,$$

the weight to be added to the arm of the balance from which the brass is suspended, to restore the equilibrium.

(The specific gravity of standard-gold is taken = 17.2, that of water being = 1).

120. We have therefore a method of detecting the adulteration of precious metals. If a real guinea and a counterfeit one of copper and gold have the same weight when placed in a pair of scales, let them be weighed in water, when it will be found that the counterfeit will lose more of its weight than the unadulterated one. For since the specific gravity of gold exceeds that of copper, and the absolute weights of the coins are equal, the counterfeit guinea must be greater in bulk than the real one, and must therefore displace a greater quantity of water, that is, lose a greater part of its weight.

121. Hence also may be resolved the problem proposed to Archimedes by King Hiero, who having employed a goldsmith to make him a crown of gold, suspected that silver had been mixed, and enquired of Archimedes, if his suspicions could be verified or disproved without injuring the crown. The mode of detecting the fraud is not known; but it seems probable that it was done in the following manner:

A quantity of gold of the same absolute weight as the crown would evidently have the same bulk, if the crown were pure gold; but greater, if adulterated. By weighing therefore the quantity of gold and the crown in water, Archimedes would find that the crown lost more of its weight than the gold, and might therefore conclude, that as the crown must have displaced a greater portion of water than the piece of gold, its bulk must likewise have been greater; and the metal therefore adulterated, of which it was composed.

122. This method, however, would not be sufficient, if the kinds of metal were unknown: for a mixture might be made, of gold and copper, for instance, of the same weight and magnitude as a mixture of gold and silver. Also if the crown contained more than two kinds of metal, as gold, silver, and copper, the problem would be indeterminate; for the three metals might be combined in several ways such that the mixture should have the same weight and magnitude: and *a fortiori* if there was a greater number of metals.

Prob. Given the specific gravities of two bodies; to find the ratio of their quantities of matter, when they balance each other in a fluid whose specific gravity is also given.

If A and B be the magnitudes, a and b the specific gravities of the bodies respectively, and s the specific gravity of the fluid,

the weight of the magnitude A *in* the fluid $= A \,.\, (a-s)$,

the weight of............ B *in* the fluid $= B \,.\, (b-s)$.

$$\text{But } A \,.\, (a-s) = B \,.\, (b-s);$$

$$\therefore \; A : B :: b-s : a-s,$$

$$\text{and } Aa : Bb :: a \,.\, (b-s) : b \,.\, (a-s).$$

Prob. A cone A, and a cylinder B, of the same specific gravity, base, and altitude, balance each other at the extremities of a straight lever, when immersed in a fluid of given specific gravity. Supposing a cone $= A$ cut out of B, and its place supplied by another of half its specific gravity; find what part must be cut off from A to restore the equilibrium.

Let A = the magnitude of the cone;

$\therefore\; 3A$ = that of the cylinder.

Let s = the specific gravity of the cone and cylinder,

and s' = that of the fluid;

$\therefore\ 3A.(s-s')=$ the weight of the cylinder in the fluid,

and $A.(s-s')=$ the weight of the cone in the fluid,

whence the arms of the lever are in the proportion of 3 : 1.

Suppose $x=$ the part remaining, when a portion is cut off from A;

$\therefore\ x.(s-s')=$ weight of the remaining part of the cone.

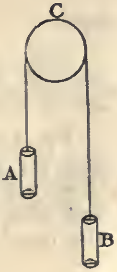
Also $2As+A\times\frac{1}{2}s-3As'=$ weight of the cylinder on the second supposition;

$$\therefore\ 3x.(s-s')=(2As+\tfrac{1}{2}As-3As')\times 1;$$

$$\text{whence } x=\frac{5s-6s'}{6.(s-s')}.A,$$

$$\text{and } \therefore \text{ the part cut off} = \frac{A}{6}\times\frac{s}{s-s'}.$$

Prob. Two equal cylinders A and B, whose density is $(2s)$ and altitude (a), are immersed in different fluids *viz.* A in a fluid whose density varies as the depth, and B in a fluid of the uniform density (s). When connected by a string passing over a fixed pulley, they balance in a given position. Supposing B depressed through a given space, by how much must it be lengthened to restore the equilibrium?



The tension of the string CA is equal to the relative gravity of A, that is, to the weight of A *minus* the weight of an equal bulk of fluid.

Let $b=$ the base of either cylinder,

$c=$ the distance of the top of A from the surface when in equilibrio at first,

$r=$ the depth at which the density is s;

$\therefore\ \frac{sx}{r}=$ the density at the depth x.

The weight of A or B *in vacuo* $=2abs$, and the weight of the bulk A of fluid

$$=b\int\frac{sx}{r}\times dx,$$

(taken between the values $x=c$, and $x=a+c$,)

$$= \frac{bs}{r} \cdot \frac{(a+c)^2 - c^2}{2}$$

$$= \frac{bs}{r} \cdot \frac{2ac + a^2}{2};$$

$$\therefore \text{ the tension of } AC = 2abs - \frac{bs}{r} \cdot \frac{2ac+a^2}{2}.$$

Now if by the depression of B, c becomes c', the tension of AC becomes $2abs - \frac{bs}{r} \cdot \frac{2ac'+a^2}{2}$, which is greater than the former by $\frac{bas}{r} \cdot (c - c')$.

The tension of CB in the first position, when the altitude of the cylinder $B = a$, was $2bas - bas = bas$,
and the increase of tension $= ba's - bas = bs \cdot (a' - a)$,

$$\text{which must equal } \frac{bas}{r} \times (c - c'),$$

$$\text{hence } a' - a = \frac{a}{r} \cdot (c - c'),$$

that is, the increase of the length of $B = \frac{a}{r} \times$ the increase of depth.

123. By weighing a solid *in vacuo*, and in a fluid, the specific gravities of the fluid and solid may be compared.

For knowing the weight of the solid *in vacuo*, and in a fluid, the difference of these weights, that is, the weight lost by the solid in the fluid is known; and (110)

the weight lost : the whole weight :: the specific gravity of the fluid : that of the solid,

in which proportion, the former ratio being known, the latter is also determined.

Ex. A piece of metal whose weight is 88 grains, when immersed in water weighs only 77 grs.;

$\therefore$ its specific gravity : that of water :: 88 : 11 :: 8 : 1.

124. By weighing several solids in the same fluid, the ratio of their specific gravities may be determined.

$$\text{For (110)} \quad W' : W :: s' : s,$$

$$\therefore s = \frac{W \times s'}{W'} \propto \frac{W}{W'}.$$

If therefore any number of solids whose weights are A, B, C, &c. when weighed in the same fluid lose weights a, b, c, &c. their specific gravities are as $\frac{A}{a}$, $\frac{B}{b}$, $\frac{C}{c}$, &c.

Ex. A solid whose weight is 8 grains weighs 7 in a fluid; and another of 785 weighs 685 in the same fluid; what is the ratio of their specific gravities.

The weights lost are 1 and 100 respectively;

$\therefore$ the specific gravities are as $\frac{8}{1} : \frac{785}{100} :: 160 : 157$.

125. By weighing the same solid in several fluids, the ratio of their specific gravities may be determined.

$$\text{For} \quad W' : W :: s' : s,$$

therefore in this case $W' \propto s'$.

If therefore a solid when successively immersed in different fluids loses weights a, b, c, &c. the specific gravities of the fluids will be proportional to a, b, c, &c.

Ex. A solid whose weight is 250 grains weighs 147 in one fluid, and 120 in another. What is the ratio of their specific gravities?

The weights lost are 103 and 130;

$\therefore$ the specific gravities are as $103 : 130$.

126. The specific gravity of a solid, may be ascertained, from weighing it in air, and in a vessel filled with water.

If $W =$ the weight of the body in air, and $w =$ the weight of the vessel when filled only with water, and $w' =$ the weight of the vessel containing the solid and water; then

the weight of the quantity of water displaced $= w' - w$,

and therefore the weight lost $= W - w' + w$,

whence $W - w' + w : W :: 1 :$ the specific gravity (s);

$$\therefore s = \frac{W}{W - w' + w}.$$

This is on the supposition that the weight of the air is neglected, and the specific gravity of water $= 1$.

127. Cor. 1. If the substance is soluble in water, as is the case with many of the salts; alcohol must be used, or some other fluid, such as the essential oils, which are not capable of dissolving it: and their specific gravity being known, that of the salt may be determined.

128. Cor. 2. If the solid imbibes water without dissolving or decomposing it; let the body be weighed when perfectly dry (and call its weight W), and again when it has imbibed as much water as possible. Let the difference of these weights $= W'$; then W' is the weight of the water imbibed. Find how much water the body displaces, which suppose $= a$; the apparent specific gravity $= \frac{W}{a}$, since it has displaced a. But in order to know the specific gravity of the solid parts of the body, which do not admit water, such as the real fibrous part of sponges, we must consider that the real quantity of water displaced is not a, but $a - W'$, and $\therefore \frac{W}{a - W'}$ is the real specific gravity, neglecting the weight of the air.

129. To find the specific gravity of a solid specifically lighter than the fluid in which it is weighed.

Take a body specifically heavier than the fluid, and sufficient to make the compound bodies sink, and weigh it in the fluid; then attach to it the solid whose specific gravity is required, and weigh the mass in the fluid. Let w' = the difference between the weights of the heavier solid in the fluid and the mass in the fluid: then w' = the force of ascent of the lighter solid = the weight of a quantity of fluid of the same magnitude with the lighter solid – the weight (w) of the lighter solid *in vacuo*; whence the weight of a quantity of fluid of the same magnitude with the lighter solid $= w + w'$. Hence (8) $w + w' : w ::$ the specific gravity of the fluid (s) : that of the solid (s');

$$\therefore \; s' = s \, . \frac{w}{w+w'}.$$

130. To find the specific gravity of a body in the state of powder.

Take a metal box or quantity of wax of such a weight, that when the powder is inclosed therein, the mass may be specifically heavier than the fluid: and proceed as in the last Article.

131. If the specific gravity of air be called s, that of water being 1, and W the weight of any body in air, and W' its weight in water; then $W + s \, . \, (W - W')$ is its weight *in vacuo* very nearly.

Let w = the real weight of the body,

$$\text{then (113)} \quad w - W : w - W' :: s : 1;$$

$$\text{whence } s \, . \, (w - W') = w - W,$$

$$\text{or } w \, . \, (1 - s) = W - s W';$$

$$\therefore \; w = \frac{W - sW'}{1 - s} = W + s \, . \, (W - W') + s^2 \, . \, (W - W') + \&\text{c}.$$

and s being small, s^2 will be small compared with s, and

$$\therefore \; w = W + s \, . \, (W - W') \text{ very nearly.}$$

132. If σ be the specific gravity of a body, ascertained by weighing it in air and water, and s the specific gravity of the air at the time when the experiment was made; the correct specific gravity or that which would have been found if the body had been weighed in a vacuum instead of air, is $\sigma + s.(1-\sigma)$.

As in the last Art. $w = \dfrac{W - sW'}{1-s}$.

But the real specific gravity $\sigma' = \dfrac{w}{w - W'}$,

$$\text{and } \therefore\ \sigma' = \frac{W - sW'}{1-s} \div \left(\frac{W - sW'}{1-s} - W'\right) = \frac{W - sW'}{W - W'}.$$

$$\text{Also } \sigma = \frac{W}{W - W'};$$

$$\therefore\ W\sigma - W'\sigma = W,$$

$$\text{and } W' = W.\frac{\sigma - 1}{\sigma},$$

$$\therefore\ \sigma' = \frac{W - sW.\dfrac{\sigma-1}{\sigma}}{W - W.\dfrac{\sigma-1}{\sigma}} = \frac{\sigma - s.(\sigma-1)}{\sigma - (\sigma - 1)} = \sigma - s.(\sigma - 1).$$

133. Given the weight of a bottle when full of air, and when full of water; the weight (in air) of a solid introduced into it, and the weight of the bottle with the solid in it, when filled up with water. To determine the specific gravity of the solid, taking into consideration the weight of the air, its specific gravity at the time being n.

Let $w =$ the weight of the bottle and water,
$w' =$ the weight of the solid in air,
$x =$ the magnitude of the solid,
and $s =$ its specific gravity;

$$\therefore\ xs - xn = w', \ (115)$$

$$\text{and}\quad x = \frac{w'}{s-n};$$

$$\therefore\ \text{the weight of the solid } \textit{in vacuo} = \frac{w's}{s-n}.$$

Let w'' = the weight of the bottle with the solid in it, when filled up with water;

$\therefore\ w'' - w =$ weight of the solid − weight of equal bulk of water, whence the weight of equal bulk of water

$$= \frac{w's}{s-n} - w'' + w$$

$$= \frac{s.(w+w'-w'') - n.(w-w'')}{s-n};$$

$$\therefore\ s : 1 :: \frac{w's}{s-n} : \frac{s.(w+w'-w'') - n.(w-w'')}{s-n};$$

$$\text{whence}\quad s = \frac{w' + n.(w-w'')}{w+w'-w''}.$$

134. Cor. Let $n = 0$; $\therefore\ s = \dfrac{w'}{w+w'-w''}$, which agrees with what was shewn above,

and therefore the correction will be $\dfrac{n.(w-w'')}{w+w'-w''}$.

135. If a body float on a fluid, the centres of gravity of the body and of the fluid displaced, must, when the body is at rest, be in the same vertical line.

For the action downwards of any portion must be equal to the re-action upwards of the fluid below. But the effect downwards is the same as if the whole effect took place at the centre of gravity; the effect therefore of the re-action of the fluid under this portion must be the same as if it took place at the centre of gravity of that portion, that is, if the body floats, as if it took place at the centre of gravity of the fluid Art. 77
displaced. If then this action of the fluid upwards against

the body in a vertical line, does not pass through the centre of gravity of the body, it must give it a rotatory motion.

136. In order that a floating body may remain at rest, two conditions are necessary;

1. That the weight of the body be equal to that of the fluid displaced;

2. That the centres of gravity of the body and the fluid displaced be in the same vertical line.

For if the first condition fail, the body will have a progressive motion in a vertical direction: if the second, it will turn round its centre of gravity, till this centre and that of the fluid displaced fall in the same vertical. Where both conditions fail, the body will at the same time have a progressive and rotatory motion.

137. Cor. 1. Hence every homogeneous plane figure, divided into two equal and similar parts by an axis supposed to be vertical; and every homogeneous solid generated by the revolution of a curve round a vertical axis will remain in equilibrio in that position in a fluid of greater specific gravity.

For the weight of the body being specifically less than that of the fluid will be supported by the pressure upwards of the fluid (107); and the centres of gravity of the whole and of the part immersed will evidently be in the same vertical line*.

138. Cor. 2. Every homogeneous prismatic body, whose axis is horizontal, will remain in equilibrio on a fluid when the centre of gravity of the section made through its middle, parallel to the bases, is in the same vertical line with that of the section which is immersed.

For the centres of gravity of the prism and of the part immersed may be considered as being in those points; and the prism is then in equilibrio.

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* It may be observed that the converse is not generally true, that is, if a homogeneous body divided into symmetrical parts by its axis is in equilibrio in a fluid, it does not follow that its axis is vertical.

139. Def. The vertical line passing through the two centres of gravity of the solid and of the part immersed is called *the line of support*.

140. When a body floats, the magnitude of the part immersed is to the magnitude of the solid, as the specific gravity of the solid is to that of the fluid.

For since the solid floats in equilibrio, its weight (101) is equal to the weight of a quantity of fluid of the same magnitude as the part immersed. If therefore M and m be the magnitudes of the solid and the part immersed, and S and s the specific gravities of the solid and fluid,

$$MS = ms,$$

$$\text{and } m : M :: S : s.$$

1. Let a solid, whose specific gravity $= 4$, float in a fluid whose specific gravity $= 10$. Required the proportion of the parts immersed and extant.

$$\text{Here } m : M :: 4 : 10;$$

$$\therefore m = \tfrac{2}{5} M,$$

$$\text{and the parts are as } \tfrac{2}{5} : \tfrac{3}{5} :: 2 : 3.$$

2. A cubic foot of wood, whose specific gravity is to that of water as 48 : 52 is put in water. How much will be immersed; and what will make it level with the water.

$$\text{Here } m : M :: 48 : 52,$$

$$\therefore m = \tfrac{12}{13} M;$$

$$\text{and } 48x + 48M = 52M,$$

$$\text{whence } x = \tfrac{1}{12} M;$$

or there must be placed on it a portion equal to $\frac{1}{12}$ of the original magnitude.

3. A cubical iceberg is 100 feet above the level of the sea, its sides being vertical. Given the specific gravity of sea-water $= 1.0263$ and of ice $= 0.9214$ at the temperature of 32°; to find its dimensions.

If $x =$ the length of one side,

$x - 100 =$ the length of the piece under water;

$$\therefore\ x^3 - 100x^2 : x^3 :: 0.9214 : 1.0263,$$

$$\text{and}\quad 100 : x :: 0.1049 : 1.0263;$$

$$\therefore\ x = \frac{102.63}{0.1049} = 978.3 \text{ feet};$$

$$\text{and } x^3 = 936302451.687.$$

141. Cor. 1. If two bodies float on the same fluid, their specific gravities will be directly as the parts immersed, and inversely as their whole magnitudes.

142. Cor. 2. If the same solid float on different fluids, the specific gravities of the fluids will be inversely as the parts immersed. For in this case, the magnitude and specific gravity of the solid being given, either of the extreme terms of the proportion will vary inversely as the other.

Ex. If the same solid float on two fluids, and in one of them $\frac{1}{5}$th of the solid be above the surface, and in the other $\frac{1}{7}$th, the specific gravities of the fluids will be as $\frac{5}{4} : \frac{7}{6} :: 15 : 14$.

143. Cor. 3. If different bodies float on different fluids, and the parts immersed be equal; the specific gravities of the fluids will be directly as the magnitudes of the bodies and their specific gravities jointly, or directly as their weights.

$$\text{For } s = \frac{MS}{m} \propto MS \propto W.$$

144. Cor. 4. If bodies of the same weight, but of different specific gravities float on the same fluid, the magnitudes of the parts immersed are equal.

145. Cor. 5. If the magnitude of the part immersed be increased or diminished by any quantity, the absolute weight of the body must be increased or diminished by a weight equal to the weight of the portion of fluid which is more or less displaced than before.

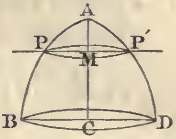
If $m' =$ the additional magnitude immersed,

then $(m + m') . s =$ the weight of the whole fluid displaced,

$$\text{and } \therefore\ (m + m') . s = MS + W'.$$

Now $MS = ms$; $\therefore$ $W' = m's$, which is the weight of the additional fluid displaced.

PROB. To find how deep a paraboloid will sink in a fluid, whose specific gravity is n times that of the solid.



Suppose PMP' the section made by plane of floatation;

$$\text{then } PBDP' : ABD :: 1 : n;$$

$$\therefore\ APP' : ABD :: n-1 : n,$$

$$\text{or } x^2 : a^2 :: n-1 : n;$$

$$\text{if } AC = a, \text{ and } AM = x;$$

$$\therefore\ x : a :: n\sqrt{n-1} : \sqrt{n},$$

$$\text{and } a - x : a :: \sqrt{n} - \sqrt{n-1} : \sqrt{n},$$

$$\text{whence } MC = a - x = a \,.\, \frac{\sqrt{n} - \sqrt{n-1}}{\sqrt{n}}.$$

PROB. A cubical inch of metal, whose specific gravity is to that of water :: m : 1, is formed into a hollow cone, and immersed with its vertex downwards. Determine the ratio of the exterior diameter of its base to the altitude, when the surface immersed is a minimum.

Let r = the radius of the base of the cone,

x = the perpendicular altitude,

z = the depth to which it is immersed in the water with its vertex downwards.

Then $x : r :: z$: the radius of the section of the cone made by the plane of floatation, which $\therefore = \frac{rz}{x}$, and the weight of the water displaced $= \frac{\pi r^2 z^3}{3x^2} \times 1 = 1 \times m$, the weight of the metal;

$$\therefore\ z^3 = \frac{3mx^2}{\pi r^2},$$

$$\text{and } z = \sqrt[3]{\frac{3m}{\pi r^2}} \,.\, x^{\frac{2}{3}}.$$

Now as the surface of the whole cone $= \pi r \sqrt{r^2 + x^2}$, and similar surfaces are to each other as the squares of like sides;

$$\therefore x^2 : z^2 :: \pi r \sqrt{r^2 + x^2} : \text{surface immersed} = \frac{\pi r z^2 \sqrt{r^2 + x^2}}{x^2};$$

$$\text{whence} \quad \pi r . \sqrt[3]{\frac{9 m^2}{\pi^2 r^4}} \times \frac{x^{\frac{4}{3}} \sqrt{r^2 + x^2}}{x^2} = \text{min.}$$

$$\therefore \frac{r^2 + x^2}{x^{\frac{4}{3}}} = \text{min.}$$

$$\text{whence} \quad 2x\,dx \,.\, x^{\frac{4}{3}} = \tfrac{4}{3} x^{\frac{1}{3}} . (r^2 + x^2) . dx,$$

$$\text{and} \quad 2x^2 = \tfrac{4}{3} . (r^2 + x^2);$$

$$\therefore x^2 = 2r^2, \quad \text{and} \quad x = \sqrt{2} . r,$$

$$\text{or} \quad x : r :: \sqrt{2} : 1.$$

Prob. A hemispherical vessel of given weight floats upon a fluid with one-third of its axis below the surface. Required the weight which must be put into it so that it may float with $\frac{2}{3}$ of its axis below the surface.

Let $W =$ the weight of the hemisphere.

Since the content of a spherical segment $= \pi . (r x^2 - \frac{1}{3} x^3)$;

$$\therefore W = s \pi r^3 . \left(\frac{1}{9} - \frac{1}{81}\right),$$

$$\text{and} \quad s = \frac{81 W}{8 \pi r^3}.$$

$$\text{And} \quad W + W' = s \pi r^3 . \left(\frac{4}{9} - \frac{8}{81}\right)$$

$$= \pi r^3 . \frac{28}{81} . \frac{81 W}{8 \pi r^3} = \frac{7 W}{2};$$

$$\therefore W' = \frac{5 W}{2}.$$

Prob. To find the thickness of a hollow globe, made of any substance, whose specific gravity is known, so that it may swim when immersed wholly or in part in a homogeneous fluid whose specific gravity is also known.

Let D and D' be the exterior and interior diameters,
and x = the part of the vertical diameter immersed;
the content of the shell $= \frac{1}{6}\pi.(D^3 - D'^3)$;
and the magnitude of the fluid displaced $= \pi.(\frac{1}{2}Dx^2 - \frac{1}{3}x^3)$.

Let s and s' be the specific gravities of the globe and fluid,

$$\text{then } \frac{1}{6}\pi s.(D^3 - D'^3) = \pi s'.(\tfrac{1}{2}Dx^2 - \tfrac{1}{3}x^3),$$

$$\text{whence } D' = \left(\frac{2s'x^3}{s} - \frac{3s'Dx^2}{s} + D^3\right)^{\frac{1}{3}}$$

and $\frac{1}{2}.(D - D')$ is the thickness of the shell.

If we suppose the shell to swim in the fluid when wholly immersed, $D = x$, and $D' = D\sqrt[3]{1 - \frac{s'}{s}}$.

Let the sphere be copper, and its diameter = 10 feet, and let the fluid medium be air, $s = 7788$, $s' = 1.2$, and

$$D' = 10\sqrt[3]{1 - \frac{1.2}{7788}} = 9.9995\,;$$

and $\frac{1}{2}.(D - D') = 0.00025$ of a foot, or 0.003 part of an inch = the thickness of the copper when the globe swims in air.

Prob. If a globe of given diameter float on common water; to determine the area of the section at the surface of the water, when the specific gravity of the globe is to that of water :: 3 : 5.

$$\text{Here } \pi.(\tfrac{1}{2}Dx^2 - \tfrac{1}{3}x^3) : \pi.\tfrac{1}{6}D^3 :: 3 : 5,$$

$$\text{or } 3Dx^2 - 2x^3 : D^3 :: 3 : 5,$$

$$\text{and } 10x^3 - 15x^2 + 3D^3 = 0\,;$$

from which equation a value of x may be obtained; whence $\pi.(Dx - x^2)$ = the area of the section may be found.

Prob. Determine the thickness of a right-angled cone of copper which shall just float with its edge level with the surface of the fluid; the specific gravities of the copper and fluid being :: 9 : 1, and the interior and exterior surfaces having a common base.

Let $r =$ the inner radius, and $t =$ the thickness;

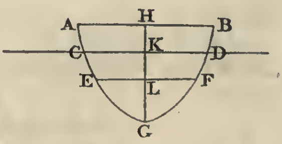
$$\text{then } \frac{\pi.(r+t)^3}{3} = 9.\frac{\pi.[(r+t)^3-r^3]}{3};$$

$$\therefore (r+t)^3 = 9.(r+t)^3 - 9r^3,$$

$$\text{and } 2.(r+t) = r\sqrt[3]{9};$$

$$\therefore t = \frac{\sqrt[3]{9}-2}{2}.r.$$

Prob. A given frustum of a paraboloid, whose density is D, sinks to a depth (a) in a fluid. How far will a similar and equal frustum similarly immersed, whose density varies inversely as the distance from the vertex, and at the smaller end $= D$, sink in the same fluid.



Let $ABFE$ be the frustum, and G the vertex of the parabola.

Let $KL = a$, $LH = b$, $GL = c$;

$$\text{then } GEF : GCD : GAB = c^2 : (c+a)^2 : (c+b)^2;$$

$$\therefore ECDF : EABF :: (c+a)^2 - c^2 : (c+b)^2 - c^2$$

$$:: 2ac + a^2 : 2bc + b^2.$$

Now if the specific gravity of the fluid $= S$,

$$S : D :: EABF : ECDF$$

$$:: 2bc + b^2 : 2ac + a^2;$$

$$\therefore S = D.\frac{b+2c}{a+2c}.\frac{b}{a}.$$

If the density $\propto \frac{1}{\text{dist.}}$, and at distance $c = D$, at the distance x it will $= D.\frac{c}{x}$;

$$\therefore \text{ the weight} = \int \pi y^2 . \frac{Dc}{x} . dx = \int \pi l x . D . \frac{c}{x} . dx$$

$$= \int \pi l D c . dx,$$

l being = the latus rectum;

$$\therefore \text{ the weight of the frustum} = \pi l D c b.$$

And if x = the depth that this frustum sinks, the weight of a quantity of fluid of the same magnitude as the part immersed is

$$\tfrac{1}{2} \pi l . [(c+x)^2 - c^2] . S = \tfrac{1}{2} \pi l . (2cx + x^2) . D . \left(\frac{b+2c}{a+2c}\right) . \frac{b}{a},$$

which is equal to the whole weight of the frustum, $= D . \pi l c b$;

$$\therefore \frac{2cx + x^2}{2} . \frac{b+2c}{a+2c} . \frac{1}{a} = c;$$

$$\text{or } 2cx + x^2 = 2ac . \frac{a+2c}{b+2c},$$

from the solution of which equation x may be found.

Prob. If there be two bodies A and B, whose specific gravities are s and s', one greater and the other less than that of water, and the weight of A is W; to determine what weight (w) of B must be annexed to A so that the specific gravity of the compound may be the same as that of water.

The magnitude of $A = \frac{W}{s}$, and the magnitude of $B = \frac{w}{s'}$;

$\therefore$ the magnitude of $A + B = \frac{W}{s} + \frac{w}{s'} = \frac{Ws' + ws}{ss'}$, which is the magnitude of water displaced by $A + B$.

Hence the weight of the water displaced $= \frac{Ws'\sigma + ws\sigma}{ss'}$, which by the hypothesis is equal to the sum of the weights;

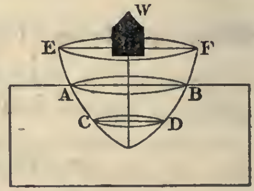
$$\text{whence } W + w = \frac{Ws'\sigma + ws\sigma}{ss'};$$

$$\therefore \quad W.(ss'-s'\sigma) = w.(s\sigma - ss'),$$

$$\text{and} \quad w = W.\frac{(s-\sigma).s'}{(\sigma-s').s}.$$

Cor. If $\sigma = 1$, $w = W.\dfrac{(s-1).s'}{(1-s').s}$.

146. If a floating body be in equilibrio, and a weight W be added, the body will sink. Supposing the centres of gravity of the fluid displaced and of the body to be in the same vertical line; to determine the depth (x) through which it is depressed.



Let $A =$ the area of the section of the body at the surface of the water, when it has sunk by the addition of W; it may be determined in terms of x from its figure and position before W was added; therefore $\int A\,dx$ will be the magnitude $ABDC$ of the body which is immersed in addition; whence if $s =$ the specific gravity of the fluid, $s\int A dx =$ the weight of the fluid displaced (101). Hence $W = s\int A\,dx$, in which the value of A being substituted, x may be determined.

147. Cor. 1. If the body be a solid of revolution, and $y =$ the radius of the section AB, $A = \pi y^2$;

$$\therefore W = \pi s \int y^2\,dx.$$

Ex. Suppose the body to be a paraboloid with its axis vertical, and in equilibrio; $y^2 = m.(a+x)$ where $m =$ the parameter, and $a =$ the distance of the vertex from the surface of the fluid;

hence $W = \frac{1}{2} m s \pi .(a+x)^2 + C = \frac{1}{2} m s \pi . [(a+x)^2 - a^2]$;

$$\therefore x = \pm \sqrt{\frac{2W}{ms\pi} + a^2} - a$$

Here the negative sign answers to the case where W would be taken away.

148. Cor. 2. The preceding formula (146) is applicable to prisms and right cylinders with any base, if the generating plane be vertical or horizontal. In the former case A is the area of a rectangle formed by the intersection of the cylinder with the surface of the fluid in the second state of equilibrium. And in the other case A is the constant area of the base, whence $W = sAx$; $\therefore\ x = \dfrac{W}{sA}$.

In the case of a right cylinder with a parabolic base whose generating plane is vertical, and whose length $= h$,

$$y^2 = m\,.\,(a + x);$$

$$\therefore\ A = 2h\sqrt{m}\,.\,\sqrt{a+x},$$

$$\text{and}\quad W = \frac{4sh\sqrt{m}}{3}\,.\,\{(a+x)^{\frac{3}{2}} - a^{\frac{3}{2}}\},$$

whence x may be determined.

If this cylinder has its generating plane horizontal, the base being the segment of a parabola, whose area $A = \frac{2}{3}cd$, c and d being the two greatest dimensions of the segment,

$$x = \frac{3W}{2scd}.$$

149. Cor. 3. The same may be applied in a similar manner to the subtraction of a weight W.

150. Cor. 4. If a body symmetrical with respect to its vertical axis floats on a fluid; to determine the weight W which must be placed on it to make it just sink to the surface.

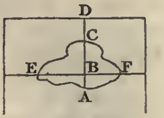
If AB be the section of the body coincident with the surface of the fluid before the addition of W, and $x =$ the distance between AB and EF; $W =$ the weight of a quantity of fluid (146) whose magnitude is $EABF$, $= s\int A\,dx + C$.

Or, if $M =$ the whole magnitude, and s and s' be the specific gravities of the fluid and solid,

$$W + Ms' = Ms;$$

$$\therefore\ W = M\,.\,(s - s').$$

151. If in two fluids which do not mix, a solid be immersed specifically heavier than the first and lighter than the second; it will float between them when the weights of the fluids respectively displaced are together equal to the weight of the solid.



Since the solid cannot wholly rest in either fluid (107), it rests between both in some position. Let *EF* be the common surface of the two fluids, and *D* the upper surface of the lighter. Now the pressure downwards on any point *A* of the base is the weight of the incumbent line of solid particles *AC*, + the weight of the superincumbent column of fluid *CD*. And the pressure upwards against the same point is the weight of a column of heavier fluid whose magnitude is *AB*, + the weight of a column of the lighter whose magnitude is *BD*. Hence the difference between the pressures downwards and upwards is equal to the difference between the weight of the solid column *AC*, and the weights of two columns *AB* and *BC* of the heavier and lighter fluids respectively. Hence the difference between the whole pressures downwards and upwards is equal to the difference between the weight of the solid, and the weights of portions of the two fluids, whose magnitudes are *AEF*, *FEC*.

Prob. If s and s' are the specific gravities of the two fluids; what must be the specific gravity of the body so that $\frac{1}{n}$th part of it may be in the lighter fluid.

Let n denote the magnitude of the body;

$\therefore$ $1 =$ the part in the lighter, and $n - 1 =$ the part in the heavier fluid,

$$\text{and } n\sigma = s' \times 1 + s\,.\,(n-1);$$

$$\therefore \sigma = \frac{s' + (n-1)\,.\,s}{n}.$$

152. Cor. 1. Hence the ratio of the parts immersed in the two fluids may be determined.

Let M and M' be the magnitudes of the parts; s and s' the specific gravities of the two fluids, and σ the specific gravity of the solid,

$$\text{then } Ms + M's' = (M + M') \, . \, \sigma;$$

$$\therefore \; M \, . \, (s - \sigma) = M' \, . \, (\sigma - s'),$$

$$\text{and } M : M' :: \sigma - s' : s - \sigma.$$

153. Cor. 2. Hence $M : M + M' : \sigma - s' : s - s'$;
and $M' : M + M' :: s - \sigma : s - s'$.

154. Cor. 3. If the solid float on one fluid, $s' = 0$,

$$\text{and } M : M' :: \sigma : s - \sigma.$$

Also $M : M + M' :: \sigma : s$, which agrees with Art. (140).

155. Cor. 4. If the specific gravity of the lighter fluid be increased, the third term $(\sigma - s')$ in the proportion is diminished, the fourth remaining the same; therefore the first will be diminished compared with the second, and the part of the body in the lighter fluid will be increased.

Thus if any substance float on the surface of a fluid *in vacuo*, on admitting the air, the substance will rise higher above the surface; and the proportion of the part immersed to the whole will be diminished.

Prob. A lighter fluid rests upon a heavier, and their specific gravities are respectively 3 and 6. A paraboloid, whose equation is $a^3 x = y^4$, floats between them with its vertex downwards, and sinks with $\frac{1}{4}$th of its axis in the heavier. Determine the specific gravity of the solid.

Let M and M' be the parts respectively immersed in the heavier and lighter fluid, and S the specific gravity of the solid;

$$\text{then } 6M + 3M' = (M + M') \, . \, S.$$

And since $y^4 \propto x$, $y^2 \propto x^{\frac{1}{2}}$; $\therefore \; y^2 x \propto x^{\frac{3}{2}}$;

$$\text{and } M : M + M' :: 1 : 4^{\frac{3}{2}} :: 1 : 8,$$

$$\therefore\ M : M' :: 1 : 7,$$

$$\text{and } 6M + 21M = 8MS,$$

$$\therefore\ S = \tfrac{27}{8} = 3.375.$$

Prob. From the vertex of a paraboloid of given dimensions, a part, equal to $\frac{1}{4}$th of the whole, is cut off by a plane parallel to the base; and the frustum being then placed in a fluid with the smaller end downwards, sinks till the plane of floatation bisects the axis which is vertical. To determine the specific gravity of the paraboloid; that of the fluid, and the density of the atmosphere being given.

Let $GH = a$; $s =$ the specific gravity of the paraboloid, (see Fig. p. 90.)

$s' =$ that of the fluid,

and $s'' =$ density of the atmosphere.

Since EGF the part cut off $= \frac{1}{4} AGB$,

$$GL^2 : GH^2 :: EGF : AGB :: 1 : 4;$$

$$\therefore\ GL = \tfrac{1}{2} GH.$$

$$\text{And } AGB : CGD : EGF = a^2 : (\tfrac{3}{4}a)^2 : (\tfrac{1}{2}a)^2 = 16 : 9 : 4;$$

$$\therefore\ ACDB : CEFD :: 7 : 5;$$

$$\therefore\ 7s'' + 5s' = 12s,$$

$$\text{and } s = \frac{7s'' + 5s'}{12}.$$

Prob. A body floats on water. Find the ratio of the parts immersed, when the imcumbent fluid is air and also when it is a vacuum.

Let M' be the magnitude of the part immersed when the body floats in air and water; and $M + M'$ that of the whole body.

If s, s' s'', represent the specific gravities of air, water, and the body respectively, then (153)

$$\frac{M + M'}{M'} = \frac{s' - s}{s'' - s}.$$

Now if M'' be the magnitude of the part immersed when the body floats in a vacuum and water, $s = 0$,

$$\text{and} \quad \frac{M+M'}{M''} = \frac{s'}{s''}.$$

$$\text{Hence} \quad \frac{M''}{M'} = \frac{s''.(s'-s)}{s'.(s''-s)} = \frac{\frac{s'-s}{s'}}{\frac{s''-s}{s''}} = \frac{1-\frac{s}{s'}}{1-\frac{s}{s''}}$$

$$= \left(1 - \frac{s}{s'}\right).\left(1 + \frac{s}{s''}\right), \text{ since } s \text{ is very small}$$

$$= 1 - \frac{s}{s'} + \frac{s}{s''};$$

$$\therefore M'' = M'.\left(1 - \frac{s}{s'} + \frac{s}{s''}\right).$$

Ex. Suppose the floating body oak,

$$s'' = 925, \quad s' = 1000, \quad s = 0.2 = \tfrac{1}{5};$$

$$\therefore \frac{s}{s'} = \frac{1}{5000}, \quad \frac{s}{s''} = \frac{1}{4625};$$

$$\therefore \frac{s}{s''} - \frac{s}{s'} = \frac{5000-4625}{23125000} = \frac{375}{23125000} = 0.000016;$$

whence $M'' = M' \times 1.000016$.

156. The *Hydrometer* consists of a long stem, which is caused to rest perpendicularly to the surface of a fluid, by a ball of sufficient weight attached to it; and the stem is graduated into aliquot parts of the whole instrument.

157. In order to measure the specific gravities of fluids, let the instrument be successively immersed in two fluids, and the magnitudes of the parts immersed be observed. Then the magnitude of the part immersed in the first : that immersed in the second :: the specific gravity of the second fluid : that of the first; in which proportion the first ratio being known,

the last is also; and if either of its terms be given, the other may be ascertained.

158. Cor. 1. In order that the divisions of the stem may indicate equal changes of specific gravity, they must be in a series of harmonic progressionals.

159. Cor. 2. If the stem be divided in arithmetic progression, its diameter must increase upwards.

For if M and M' represent the parts immersed in two fluids respectively whose specific gravities are S and S',

$$S - S' \propto \frac{1}{M} - \frac{1}{M'} \propto \frac{M' - M}{MM'} \propto \frac{M' - M}{M^2},$$

when $M' - M$ is very small.

If therefore $S - S'$ be constant, which is the case when the graduation is equal, $M' - M \propto M^2$. But $M' - M$ being the portion of the stem included between two successive divisions is nearly a cylinder whose height is given, and whose base is the thickness of the stem at that place; therefore the thickness $\propto M$.

160. Cor. 3. When a body floats between two fluids with a part in each, $MS + M'S' = (M + M') \cdot \sigma$; and this is the case with the common hydrometer. Suppose it would sink to the same degree in a fluid whose specific gravity is $S + S''$, independent of the air; then S'' is the quantity to be deducted from the indication of the instrument, to attain the true result; and

$$M \cdot (S + S'') = (M + M') \cdot \sigma = MS + M'S';$$

$$\therefore\ MS'' = M'S';$$

$$\text{and } S'' = \frac{M'}{M} \cdot S'.$$

161. Cor. 4. If the pressure of the atmosphere be considered; since (153),

$$M : M + M' :: \sigma - s' : s - s';$$

$$\therefore\ M \propto \frac{1}{s - s'};$$

for if the lighter fluid and the solid be given, $M + M'$ and $\sigma - s'$ are given. Hence if the stem be cylindrical, and l and l' the lengths of the stem respectively immersed in two fluids, whose specific gravities are s' and s'', communicating with air,

$$l : l' :: s'' - s : s' - s;$$

$\therefore$ the stem may be graduated.

Prob. To determine the nature of the stem, so that the part (x) of it above the surface of the fluid may be proportional to the specific gravity of the fluid in which it is immersed.

Let a be the value of x when $S = 1$,

then $\frac{x}{a} = S = \frac{W}{M}$, if M be the part immersed;

and $\frac{dx}{a} = -\frac{W\,dM}{M^2}$.

Let B be the whole bulk of the hydrometer,

then $M = B -$ the part of the stem not immersed;

and if the stem be a solid of revolution, $M = B - \int \pi y^2 dx$;

$$\text{whence } \frac{dx}{a} = \frac{W\pi y^2 dx}{(B - \int \pi y^2 dx)^2};$$

$$\therefore\ B - \int \pi y^2 dx = \sqrt{\pi a W}\,.\,y,$$

$$\text{and } -\pi y^2 dx = \sqrt{\pi a W}\,.\,dy;$$

$$\therefore\ dx = \sqrt{\frac{aW}{\pi}}\,.\,\frac{-dy}{y^2},$$

$$\text{whence } x + C = \sqrt{\frac{aW}{\pi}} \times \frac{1}{y};$$

$$\text{and } yx + Cy = \sqrt{\frac{aW}{\pi}}$$

the equation to an hyperbola between the asymptotes.

162. A considerable improvement has been made in the hydrometer, by placing a small brass cup on the top of the stem, into which small weights may be put, so as to sink it in different fluids to the same point of the stem. In this case, the part immersed being the same, the specific gravities of the fluids will be as the whole weights (143); which are known from knowing the weight of the instrument, and the weight added.

Thus let $W =$ the weight necessary to make it sink in one fluid, and $W \pm w =$ the weight necessary to make it sink to the same point in another fluid.

Then if S and S' be the specific gravities of the fluids,

$$W : W \pm w :: S : S';$$

$$\therefore\ S' = S \,.\, \frac{W \pm w}{W} = S \,.\left(1 \pm \frac{w}{W}\right);$$

in which equation knowing W and w, and the specific gravity of one of the fluids, which will be 1.000, if that fluid is water,

$$S' = 1.000 \times \left(1 \pm \frac{w}{W}\right), \text{ is also known.}$$

163. This is nearly the construction of Nicholson's hydrometer, which is capable of ascertaining the specific gravities of bodies also.

It consists of a hollow copper ball attached to a dish by means of a stem made of hardened steel, and about $\frac{1}{40}$th of an inch in diameter. An iron stirrup fixed to the lower extremity of the ball, carries another dish sufficiently heavy to keep the instrument in a vertical position. The parts of the instrument are so adjusted, that when a given weight (e. g. 1000 grains) are placed in the upper dish, the whole will sink in distilled water, at the temperature of 60° of Fahrenheit, to the point (m) in the middle of the stem.

164. To find the specific gravity of a fluid*.

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* Since the specific gravities of fluids vary when their temperatures vary, in comparing those of different fluids, we must first reduce them

Suppose W = the weight of the hydrometer, w = the weight to sink it in a given fluid to the point (m), and w' = the weight to be added or taken away, in the fluid whose specific gravity is required. Then $W+w$ and $W+w\pm w'$ are the weights of the fluids displaced;

$$\therefore\ S : S' :: W+w\pm w' : W+w :: 1\pm\frac{w'}{W+w} : 1;$$

and one specific gravity being given, the other may be found.

165. To find the specific gravity of a solid which does not exceed 1000 grains in weight.

Let w = the weight in the upper dish necessary to make the instrument descend in distilled water to the point (m). When this weight is removed, let the body be placed in the dish, with an additional weight w' sufficient to make the instrument sink to (m). The magnitude of the fluid displaced in the two cases is the same; and the weight of the body in air $= w - w'$.

Now let the body be placed in the lower dish; and the instrument sunk to the point (m) by the addition of a weight w'', that is, to the weight w' in the former case, add the weight $w''-w'$, which is the loss of weight of the body in the water, or the weight of an equal bulk of distilled water. Hence the ratio of the specific gravities is $\frac{w-w'}{w''-w'}$, w, w', w'', being the weights placed in the dish in three cases.

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them to some one temperature as a standard. This is arbitrary: the temperature of about 60° of Fahrenheit's thermometer is convenient for experiment, since it may always be readily procured in these climates. One nearer to freezing would have some advantages, because water changes its bulk very little between 32° and 45°. But this cannot always be obtained.—An experiment determines the specific gravity only in that temperature in which the bodies are examined. Some corrections arising from the expansion and contraction of fluids in consequence of change of temperature will be pointed out in a following Section.

166. As the cylindrical stem of this instrument is only $\frac{1}{40}$th of an inch in diameter, the instrument will rise or fall nearly one inch by the subtraction or addition of $\frac{1}{10}$th of a grain. It will therefore indicate changes in weight less than $\frac{1}{20}$th of a grain, which will give the specific gravities correct to five places of figures.

Prob. In a cylinder which is $\frac{3}{4}$ths filled with water, an hydrometer is observed to rest at a certain depth. Suppose the vessel filled up with fluid of three times the specific gravity of water, and the two fluids to mix, with what weight must the instrument be loaded to make it sink to the same depth as before.

Let $4A$ = the content of the vessel,

s = the specific gravity of water,

the specific gravity of the mixture $= \dfrac{3A \,.\, s + A \,.\, 3s}{4A} = \dfrac{3s}{2}$.

Call x the additional weight, and w the weight of the instrument; then the part immersed being the same, the whole weight $\propto$ the specific gravity (143),

$$\therefore\ w : w+x :: 1 : \tfrac{3}{2},$$

$$\text{and } w : x :: 1 : \tfrac{1}{2};$$

$$\therefore\ x = \tfrac{1}{2} w.$$

167. Given the weight of a common hydrometer, and the magnitude of the lower part which is not cylindrical; to determine the length of the stem immersed in a given fluid.

Let w = the weight of the hydrometer, and m = the given magnitude; s = the specific gravity of the fluid, l = the length of the stem immersed, and r = its radius;

$$\therefore\ \pi r^2 l = \text{the content of the stem.}$$

Now the instrument being in equilibrio, will displace a portion of fluid of the same weight as the hydrometer;

$$\therefore\ w = s \,.\, (m + \pi r^2 l),$$

$$\text{whence } l = \frac{w - ms}{\pi r^2 s}.$$

168. Cor. 1. A small variation in w or s will cause a considerable one in l.

For $w - ms =$ the weight of the fluid displaced by the stem: and from the smallness of the radius of the stem, $w - ms$ is a very small quantity.

169. Cor. 2. To determine the difference of depth to which it sinks, arising from a change of density of the fluid.

Let s become s', and l be changed to l',

$$\text{then } l' = \frac{w - ms'}{\pi r^2 s'};$$

$$\therefore\ l - l' = \frac{w.(s' - s)}{\pi r^2 s s'}.$$

170. Cor. 3. The sensibility of the instrument may be increased by increasing w, the weight of the fluid displaced, or by diminishing the diameter of the stem.

171. Cor. 4. If the specific gravity remains the same, the sensibility of the instrument, or the quantity by which it sinks in consequence of the addition of a small weight $w' \propto \frac{w'}{r^2}$.

For suppose by the addition of w', l becomes $= l'$,

$$\text{then (145) } \pi r^2 . (l' - l) . s = w';$$

$$\therefore\ l' - l = \frac{w'}{\pi r^2 s} \propto \frac{w'}{r^2}.$$

172. Cor. 5. If the specific gravity be altered, or s become s'; and in order to keep the instrument at the same depth, w becomes $w + w'$,

$$s' = \frac{w + w'}{m + \pi r^2 l};$$

$$\text{but } s = \frac{w}{m + \pi r^2 l};$$

$$\therefore s' : s :: w + w' : w,$$

which agrees with what has been proved before.

173. Cor. 6. Hence $s' - s : s :: w' : w$;

$$\therefore s' - s = \frac{w'}{w} . s,$$

the difference of the specific gravities.

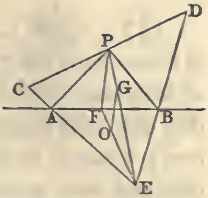
174. The *Hydrostatical Balance* consists of an inverted balance sustained on a fulcrum at the bottom of a vessel of fluid; to one end of which a body specifically lighter than the fluid is suspended by an horse-hair; and the other end connected also by an horse-hair with the extremity of a common balance *in vacuo*. To the other extremity of the latter balance a scale is appended in which is placed a weight sufficient to counteract the tendency of the body to rise in the fluid. This weight therefore measures the force of ascent, and is equal to the difference between the weights of the solid and of an equal bulk of fluid. Hence if to the above weight be added the weight of the solid *in vacuo*, the weight of an equal bulk of fluid is known.

The weight of the scale must be counterbalanced at the other arm of the balance *in vacuo*.

175. By this balance the specific gravities of fluids also may be determined.

For let W and W' be the weights to be added in order to produce the equilibrium in the fluids respectively. These will be the weights of quantities of fluids of equal magnitudes, and $\therefore$ $W : W'$ will be the ratio of their specific gravities.

176. To determine the position of equilibrium of an homogeneous triangle floating on a fluid, supposing only one angle immersed.



Let AB be the plane of floatation. Then in order that the body may be in equilibrio in the fluid (136), the weight of the triangle CDE must be equal to the weight of the fluid triangle ABE; and the centres of gravity G and O, of the two triangles CDE, ABE must be in the same vertical line GO, which must therefore be perpendicular to the surface of the fluid.

Let P and F be the points of bisection of CD and AB; take $EG=\frac{2}{3}EP$, and $EO=\frac{2}{3}EF$; G and O are the centres of gravity of the triangles ECD and ABE. And since the sides of the triangle EPF are cut proportionally in O and G, OG is parallel to PF; and since GO must be vertical, they are perpendicular to AB. Hence if PA and PB be joined, they will be equal.

Let EC, ED, EP be represented by a, b, c, respectively; let the angle $PEC = m$, $PED=n$, $EA=x$, $EB=y$; and $s : \sigma$ the ratio of the specific gravities of the solid and fluid.

Since the area of the triangle $ECD=\frac{1}{2}ab \,.\, \sin n$,

and the area of the triangle $AEB = \frac{1}{2}xy \,.\, \sin n$;

$$\therefore\ s\,.\,ab=\sigma\,.\,xy \ \ldots\ldots \ [\text{I}]$$

$$\text{Now } PB^2=PE^2+EB^2-2EB\times PE\times\cos PEB$$
$$=c^2+y^2-2cy\,.\cos n.$$

$$\text{And } PA^2=c^2+x^2-2cx\,.\cos m;$$

$$\text{whence } y^2-2cy\,.\cos n=x^2-2cx\,.\cos m;$$

$$\therefore\ \frac{s^2a^2b^2}{\sigma^2x^2}-\frac{2csab}{\sigma x}\,.\cos n = x^2-2cx\,.\cos m,$$

$$\text{or } x^4-2cx^3\,.(\cos m)+\frac{2csab}{\sigma}\,.(\cos n)\,.\,x-\frac{s^2a^2b^2}{\sigma^2}=0\,..\,[\text{II}]$$

the roots of which equation and of [I] will give the positions of equilibrium.

177. Cor. 1. Since every equation of an even number of dimensions (*Alg.* 353), which has its last term negative, has at least two real roots, one positive and the other negative, this equation has two real roots. But the other two may be real; whence in that case (*Alg.* 308) three roots will be positive and one negative. The body therefore may have three positions of equilibrium; but it cannot have more.

178. Cor. 2. If the triangle CED be isosceles, $b = a$, and $\cos n = \cos m$;

$$\therefore\ xy = \frac{sa^2}{\sigma},$$

$$\text{and } x^4 - 2c(\cos m).x^3 + \frac{2csa^2}{\sigma}.(\cos m).x - \frac{s^2a^4}{\sigma^2} = 0,$$

$$\text{or } \left(x^2 - \frac{sa^2}{\sigma}\right).\left\{x^2 - 2c.(\cos m).x - \frac{sa^2}{\sigma^2}\right\} = 0;$$

$$\therefore\ x^2 - \frac{sa^2}{\sigma} = 0,$$

$$\text{and } x^2 - 2c.(\cos m).x - \frac{sa^2}{\sigma} = 0;$$

from the former of which equations we deduce $x = a\sqrt{\frac{s}{\sigma}}$, and $y = a\sqrt{\frac{s}{\sigma}}$, and $\therefore\ y = x$, or the triangle AEB is isosceles also, and CD the base of the given triangle is parallel to the surface of the fluid.

The second equation gives

$$x = c.\cos m \pm \sqrt{c^2.\cos^2 m - \frac{sa^2}{\sigma}};$$

$$\text{whence } y = \frac{\frac{s}{\sigma}.a^2}{c.\cos m \pm \sqrt{c^2.\cos^2 m - \frac{sa^2}{\sigma}}}$$

$$= c\,.\cos m \mp \sqrt{c^2.\cos^2 m - \frac{s\,a^2}{\sigma}},$$

which will give the other positions, when x and y are real, and x less than a, and y less than b, that is, in this case less than a.

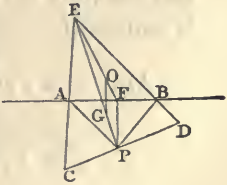
Now in order to satisfy these conditions $\frac{s\,a^2}{\sigma}$ must be less than $c^2\,.\cos^2 m$, or $\frac{s}{\sigma}$ less than $\frac{c^2}{a^2}\,.\cos^2 m$; and a must be greater than $c\,.\cos m + \sqrt{c^2\,.\cos^2 m - \frac{s\,a^2}{\sigma}}$, or $\frac{s}{\sigma}$ must be greater than $\frac{2\,a\,c\,.\cos m - a^2}{a^2}$.

When $\therefore\ \sigma = 1$, the limits of s are between $\frac{c^2\,.\cos^2 m}{a^2}$ and $\frac{2\,a\,c\,.\cos m - a^2}{a^2}$.

179. Cor. 3. If the triangle be equilateral $c\,.\cos m = \frac{3}{4}a$; and therefore the triangle besides the situation of equilibrium indicated by the first equation may have two others, if s be less than $\frac{9}{16}$, and greater than $\frac{8}{16}$, that is, if s be between the values of $\frac{9}{16}$ and $\frac{8}{16}$.

180. To determine the position of equilibrium of an homogeneous triangle, floating on a fluid, supposing two angles are immersed.

Here the centres of gravity of ECD, $ABDC$, AEB are in the same straight line; but the centres of gravity of ECD, $ABDC$ are in the same vertical; whence the centres of gravity of ECD, AEB are also in the same vertical.



As before, therefore, take $EG = \frac{2}{3} EP$ and $EO = \frac{2}{3} EF$; then OG being joined, is parallel to FP; and FP is perpendicular to AB; therefore as before, $PB = PA$.

$$\text{Now since } ECD : ACDB :: \sigma : s,$$

$$ECD : EAB :: \sigma : \sigma - s;$$

$$\therefore \; EAB = ECD \, . \frac{\sigma - s}{\sigma};$$

$$\text{whence (as in the last) } xy = \frac{\sigma - s}{\sigma} . \, ab,$$

$$\text{and } x^4 - 2cx^3 . (\cos m)$$

$$+ 2cab . \frac{\sigma - s}{\sigma} . (\cos n) . x - \left(\frac{\sigma - s}{\sigma}\right)^2 . a^2 b^2 = 0,$$

which is similar to the former equation. The body therefore may have three positions of equilibrium, but it cannot have more.

181. Cor. 1. If the triangle be isosceles, $a = b$, $\cos m = \cos n$, and the equation becomes

$$x^4 - 2c . (\cos m) . x^3 + 2c . \frac{\sigma - s}{\sigma} . (\cos m) . a^2 x - \left(\frac{\sigma - s}{\sigma}\right)^2 . a^4 = 0,$$

$$\text{whence } x^2 - \frac{\sigma - s}{\sigma} . a^2 = 0;$$

$$\text{and } x^2 - 2c . (\cos m) . x + \frac{\sigma - s}{\sigma} . a^2 = 0;$$

$$\therefore \; x = a \sqrt{\frac{\sigma - s}{\sigma}} = y,$$

and therefore the triangle is in equilibrio when the base CD is horizontal.

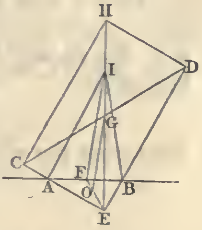
$$\text{Also } x = c . \cos m \pm \sqrt{c^2 . \cos^2 m - a^2 . \frac{\sigma - s}{\sigma}},$$

$$y = c . \cos m \mp \sqrt{c^2 . \cos^2 m - a^2 . \frac{\sigma - s}{\sigma}};$$

which give the other positions, as before, when $\frac{s}{\sigma}$ is greater than $\frac{a^2-c^2 \,.\, \cos^2 m}{a^2}$ and less than $\frac{2a^2-2ac \,.\, \cos m}{a^2}$.

182. Cor. 2. If *ECD* be equilateral, $c \,.\, \cos m = \frac{3}{4}a$; whence $\frac{s}{\sigma}$ must be greater than $\frac{7}{16}$ and less than $\frac{8}{16}$.

183. To determine the position of equilibrium of an homogeneous rectangle with one angle immersed in the fluid.



Let *AB* be the plane of floatation: bisect *AB* in *F*; join *EF*, and take $EO = \frac{2}{3}EF$; *O* is the centre of gravity of *AEB*. Now the centre of gravity of the rectangle is in *G* the intersection of the diagonals; therefore *GO* is vertical, and perpendicular to the surface of the fluid *AB*. Let $GI = \frac{1}{2}EG$. Join *IF*; it is parallel to *GO*, and therefore perpendicular to *AB*; whence $IA = IB$.

Let $EC = a$, $ED = b$, $EA = x$, $EB = y$; and let $s : \sigma$ be the ratio of the specific gravities.

Then $sab = \frac{1}{2}\sigma \,.\, xy$,

$$\text{and } IA^2 = IE^2 + EA^2 - 2AE \times IE \times \cos IEA$$

$$= IE^2 + EA^2 - 2AE \times IE \times \frac{EC}{EH}$$

$$= IE^2 + x^2 - \frac{3ax}{2}.$$

$$\text{Also } IB^2 = IE^2 + y^2 - \frac{3by}{2};$$

$$\text{whence } x^2 - \frac{3ax}{2} = y^2 - \frac{3by}{2},$$

$$\text{and } x^4 - \frac{3a}{2} \,.\, x^3 + \frac{3sab^2}{\sigma} \,.\, x - \frac{4s^2}{\sigma^2} \,.\, a^2b^2 = 0,$$

which will give the positions of equilibrium.

184. Cor. Suppose the figure to be a square, and $\therefore\ a = b$; the equation becomes

$$x^4 - \frac{3a}{2} \cdot x^3 + \frac{3sa^3}{\sigma} \cdot x - \frac{4s^2}{\sigma^2} \cdot a^4 = 0;$$

$$\text{whence } x^2 - \frac{2s}{\sigma} \cdot a^2 = 0,$$

$$\text{and } x^2 - \frac{3a}{2} \cdot x + \frac{2s}{\sigma} \cdot a^2 = 0.$$

$$\text{Hence } x = a\sqrt{\frac{2s}{\sigma}}, \quad \text{and } y = a\sqrt{\frac{2s}{\sigma}} = x,$$

and the square is in equilibrio when the diagonal CD is horizontal.

Also from the second equation, $x = \frac{a}{4} \cdot \left(3 \pm \sqrt{9 - 32 \cdot \frac{s}{\sigma}}\right)$,

$$\text{and } y = \frac{a}{4} \cdot \left(3 \mp \sqrt{9 - 32 \cdot \frac{s}{\sigma}}\right),$$

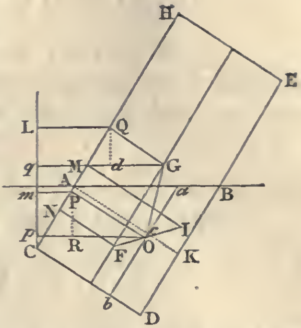
from which two positions of equilibrium are obtained when $\frac{s}{\sigma}$ is less than $\frac{9}{32}$ and greater than $\frac{8}{32}$.

185. Cor. 2. In a manner similar to the proposition, the position of equilibrium may be determined when three angles of the rectangle are immersed; by inverting the figure; whence, as in the former cases, it may be shewn that

$$\tfrac{1}{2}xy = \frac{\sigma - s}{\sigma} \cdot ab,$$

$$\text{and } x^2 - \frac{3ax}{2} = y^2 - \frac{3by}{2}.$$

186. To determine the positions of equilibrium of an homogeneous rectangle, when two angles are immersed in the fluid.



From A, where the line of floatation meets the side CH, draw AK perpendicular to the opposite side; and from C draw the vertical line CL. Let I, F, O, G be the centres of gravity of ABK, $ACDK$, $ACDB$ and $CDEH$ respectively. Draw IM, FN, OP, GQ perpendicular to CH; and Op, Gq, Pm, QL parallel to the horizon.

Let $CD=a$, $CH=b$, $CA=x$, $DB=y$, $AB=z$; and let $s:\sigma$ be the ratio of the specific gravities of the solid and fluid;

$$\text{then } s\,.\,CDEH=\sigma\,.\,CABD,$$

$$\text{or } s\,.\,ab=\sigma\,.\,\frac{a}{2}\,.\,(x+y);$$

$$\therefore\ x+y=\frac{s}{\sigma}\,.\,2b.$$

$$\text{Also } CABD\times OP=ABK\times IM+ACDK\times FN,$$

$$\text{or } \tfrac{1}{2}a\,.\,(x+y)\,.\,D=\tfrac{1}{2}a\,.\,(y-x)\,.\,\tfrac{2}{3}a+ax\,.\,\tfrac{1}{2}a,$$

$$\text{or } \frac{s}{\sigma}\,.\,ab\,.\,D=\tfrac{1}{3}a^2y+\tfrac{1}{6}a^2x;$$

$$\therefore\ D=\frac{\sigma}{s}\,.\,\frac{2ay+ax}{6b}.$$

Now the centre of gravity O is the point of bisection of aOb which is parallel to AC or BD,
and Oa or $CP=\frac{1}{2}\,.\,\{CA+ca\}$

$$=\tfrac{1}{2}x+\frac{y-x}{2a}\,.\,D, \text{ since } AK:KB::Ac:ca.$$

$$\text{Also } AB:BK::CP:Pm,$$

$$\text{or } z:y-x::\frac{x}{2}+\frac{y-x}{2a}\,.\,D:Pm;$$

$$\therefore\ Pm=\frac{y-x}{z}\,.\,\left\{\frac{x}{2}+\frac{y-x}{2a}\,.\,D\right\}.$$

Also $AB : AK :: OP : OR$,

$$\text{or} \quad z : a :: D : OR = \frac{aD}{z};$$

and $AB : BK :: CQ : QL$,

$$\text{or} \quad z : y - x :: \tfrac{1}{2}b : QL = \frac{b}{2z} \cdot (y - x).$$

Also $AB : AK :: GQ : Gd$,

$$\text{or} \quad z : a :: \tfrac{1}{2}a : Gd = \frac{a^2}{2z}.$$

Now by the second condition of equilibrium, $Op = Gq$, whence

$$aD + (y - x) \cdot \left\{\frac{x}{2} + \frac{y - x}{2a} \cdot D\right\} = \frac{a^2}{2} + (y - x) \cdot \tfrac{1}{2}b,$$

$$\text{or} \quad D \cdot [2a^2 + (y - x)^2] = a^3 + a \cdot (b - x) \cdot (y - x);$$

hence if $s' = \dfrac{s}{\sigma}$,

$$\frac{2a \cdot (2s'b - x) + ax}{6s'b} \cdot \{2a^2 + (2s'b - 2x)^2\}$$
$$= a^3 + a \cdot (b - x) \cdot (2s'b - 2x),$$

$$\text{or} \quad \frac{4s'b - x}{6s'b} \cdot \{2a^2 + (2s'b - 2x)^2\}$$
$$= a^2 + (b - x) \cdot (2s'b - 2x);$$

$$\text{whence} \quad a^2 \cdot (s'b - x) + 2 \cdot (4s'b - x) \cdot (s'b - x)^2$$
$$= 6s'b \cdot (b - x) \cdot (s'b - x),$$

which will give the positions of equilibrium.

187. Cor. 1. Since $s'b - x$ is a factor of the equation,

$$x = s'b = \frac{s}{\sigma} \cdot b, \quad \text{whence} \quad y = \frac{s}{\sigma} \cdot b,$$

and therefore the position of equilibrium determined from hence, will be when $x = y$, or when the side is horizontal. The other part of the equation

$$x^2 - 2s'bx + 4s'^2b^2 - 3s'b^2 + \tfrac{1}{2}a^2 = 0,$$

$$\text{gives} \quad x = s'b \pm \sqrt{3b^2 . (s' - s'^2) - \tfrac{1}{2}a^2},$$
$$\text{and} \quad y = s'b \mp \sqrt{3b^2 . (s' - s'^2) - \tfrac{1}{2}a^2};$$

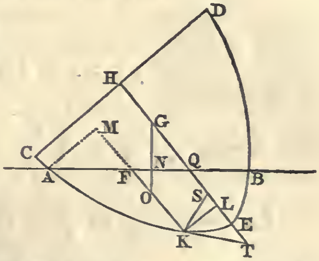
whence the rectangle will have two situations of equilibrium, provided the values of x and y are real and positive, and each less than b.

188. Cor. 2. If the rectangle be a square, $a = b$; and it is in equilibrio, when any of its sides are horizontal.

$$\text{Also} \quad x = s'a \pm a\sqrt{3 . (s' - s'^2) - \tfrac{1}{2}},$$
$$\text{and} \quad y = s'a \mp a\sqrt{3 . (s' - s'^2) - \tfrac{1}{2}};$$

whence there will be two new positions of equilibrium, when s' is between $\frac{3}{4}$ and $\dfrac{3 + \sqrt{3}}{6}$.

189. To determine the positions of equilibrium of an homogeneous parabola, supposing the extreme ordinate entirely out of the fluid.



Let EH be the axis, and CD the extreme ordinate, AB the line of floatation, which bisect in F, and draw KF parallel to EH. Let fall the perpendicular KL, and draw the tangent KT meeting HE produced in T; it is parallel to AB. Let S be the focus, and join KS. Take G and O the centres of gravity of CED and AEB. Join GO, which will be perpendicular to AB (135).

Let $EH = a$, $CH = b$, and the parameter to the axis $= \dfrac{b^2}{a} = c$, $KF = x$, $AF = y$, $KL = z$, $s =$ the specific gravity of the parabola, and $\sigma =$ that of the fluid.

Then, $EG = \frac{3}{5}a$, $KO = \frac{3}{5}x$, $LT = 2LE = \dfrac{2z^2}{c}$, $KT = \dfrac{z\sqrt{c^2 + 4z^2}}{c}$;

and from similar triangles KLT, AMF,

$$KT : KL :: AF : AM = \frac{cy}{\sqrt{c^2+4z^2}}.$$

Also the area $EDC = \frac{4}{3} EH \times HC$,

and $EAB = \frac{4}{3} AM \times FK$,

$$\text{whence } sab = \frac{\sigma c x y}{\sqrt{c^2+4z^2}} \ldots\ldots [\text{I}].$$

As the second condition of equilibrium is satisfied, if the axis be vertical, and $\therefore z = 0$,

$$sab = \sigma x y = \sigma x \sqrt{cx};$$

$$\therefore x = a \,.\, \left(\frac{s}{\sigma}\right)^{\frac{2}{3}}.$$

But if K does not coincide with E,

$$KS = SE + EL = \frac{c^2+4z^2}{4c},$$

$$y^2 = x \times 4KS = \frac{x \,.\, (c^2+4z^2)}{c},$$

$$\text{and } \therefore y = \sqrt{\frac{x}{c}} \times \sqrt{c^2+4z^2}.$$

Let this value of y be substituted in equation [I],

$$\text{and } sab = \sigma \,.\, \sqrt{cx^3},$$

$$\text{whence } x = a \,.\, \left(\frac{s}{\sigma}\right)^{\frac{2}{3}} \text{ as before.}$$

Now GO being perpendicular to AB, the triangles KLT, FNO are similar,

$$\text{whence } KT : LT :: OF : FN = \frac{4xz}{5\sqrt{c^2+4z^2}};$$

$$\therefore NQ = FQ - FN = KT - FN = \frac{z\sqrt{c^2+4z^2}}{c} - \frac{4xz}{5\sqrt{c^2+4z^2}};$$

and *KLT*, *GNQ* being similar,

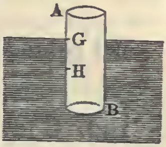
$$LT : KT :: NQ : QG = \frac{c^2 + 4z^2}{2c} - \frac{2x}{5}.$$

Now $QG = EG - EQ = EG - (QT - ET) = \frac{3}{5}a - x + \frac{z^2}{c}$;

whence $z^2 = \frac{6ac - 5c^2 - 6cx}{10} = \frac{6ac - 5c^2}{10} - \frac{6ac}{10} \cdot \left(\frac{s}{\sigma}\right)^{\frac{2}{3}}$,

whence two new positions of equilibrium will be obtained, one to the right, and the other to the left, provided $6a$ is greater than $5c + 6a \cdot \left(\frac{s}{\sigma}\right)^{\frac{2}{3}}$, or $\frac{s}{\sigma}$ less than $\left(\frac{6a - 5c}{6a}\right)^{\frac{3}{2}}$, supposed to be real and positive.

190. If a cylinder, which, when left to itself, floats vertically on the surface of a fluid, be depressed or elevated through a given depth: to determine the motion when let go.



Let the plane of floatation pass through *H* when the cylinder is at rest; and suppose it to be depressed till the point *G* is at the surface. The weight of *AB* is equal to the weight of a cylinder *BH* of the fluid (101). And when the cylinder is depressed till *G* comes to the surface, the accelerating force, which (107) is equal to the difference of the weights of the body and the fluid displaced divided by the weight of the body, is $= \frac{HG \times a \times s'}{h \times a \times s} = \frac{HG \times s'}{h \times s}$, if the height of $AB = h$, and s and s' be the specific gravities of the solid and fluid. Hence, the accelerating force varying as the distance from the position of equilibrium, the motion is the same as that of a pendulum in a cycloid whose length is $\frac{s}{s'} \cdot h$ = the altitude of *BH*;

for (force $=$) $\frac{HG \times s'}{h \times s}$: (gravity $=$) 1 :: $HG : l = \frac{sh}{s'}$,

and $\therefore$ the time of an oscillation $= \pi \sqrt{\frac{sh}{gs'}}$.

In a similar manner it may be shewn, that if the body be elevated, the force $\propto HG$, as before.

191. Cor. 1. If the cylinder has $\frac{1}{n}$th of its axis immersed when at rest; and $\frac{1}{m}$th of the part extant is suddenly taken off; to find the time of an oscillation.

Since the cylinder rests with $\frac{1}{n}$th of its axis immersed, its specific gravity is $\frac{1}{n}$th of that of the fluid (140). Now when $\frac{1}{m}$th of the part *extant* is taken off, the new position of equilibrium will be when $\frac{1}{n}$th of the remainder is immersed, that is, when $\frac{1}{n} \cdot \frac{(m-1) \cdot n+1}{mn}$ is immersed.

Hence, as before, the length of the pendulum $= \frac{(m-1) \cdot n+1}{mn^2}$,

$$\text{and the time of an oscillation} = \pi \sqrt{\frac{(m-1) \cdot n+1}{gmn^2}}.$$

192. Cor. 2. If the cylinder be depressed to A; to determine its velocity after it has risen through any altitude x; and the time of its rising.

Let $s : s' :: n : 1$; $\therefore$ $AH = (1-n) \cdot h$.

If therefore in this case AG be the height risen,

$$GH = (1-n) \cdot h - x;$$

$$\therefore \text{ the accelerating force} = \frac{(1-n) \cdot h - x}{nh},$$

$$\text{and } vdv = gFdx = g \cdot \frac{(1-n) \cdot hdx - xdx}{nh};$$

$$\text{whence } v^2 = \frac{g}{nh} \cdot \{2 \cdot (1-n) hx - x^2\}; \; C \text{ being } = 0;$$

$$\therefore v = \sqrt{\frac{g}{nh} \cdot \{2 \cdot (1-n) \cdot hx - x^2\}}.$$

Also $t = \int \frac{dx}{v} = \sqrt{\frac{nh}{g}} . \int \frac{dx}{\sqrt{2 . (1-n) . hx - x^2}}$

$$= \sqrt{\frac{nh}{g}} \times \text{arc} \left[\text{versin} = \frac{x}{(1-n) . h} \right].$$

193. Cor. 3. To determine the greatest altitude to which the body rises in the case of the last Cor.

Here $v = 0$; $\therefore 2 . (1-n) . hx - x^2 = 0$,

and $x = 2 . (1-n) . h$.

194. Cor. 4. The velocity is the greatest, when the force $= 0$, that is, when $(1-n) . h = x$, which is just half the former altitude.

195. Cor. 5. After ascending to the greatest altitude the cylinder again descends; after which it returns ascending as before; the velocity increasing to the middle point and then decreasing again to the extreme point of motion, after the manner of a pendulum oscillating. The tenacity and friction of the fluid will retard the motion, and the vibrations will decrease, till at length the cylinder will rest in the middle point of its vibration, where it will float in its quiescent state with the part $(1-n) . h$ extant.

196. Cor. 6. The greatest velocity

$$= \sqrt{\frac{g}{nh} . \{2 . (1-n) . h - x\} . x}$$

$$= \sqrt{\frac{g}{nh} . (1-n)^2 . h^2}$$

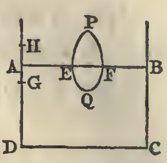
$$= (1-n) . \sqrt{\frac{gh}{n}}.$$

197. Cor. 7. The whole time of an oscillation

$$= 2 . \sqrt{\frac{nh}{g}} \times \frac{\pi}{2} = \pi \sqrt{\frac{nh}{g}};$$

which agrees with the former value. And the times of oscillation will be the same, to whatever depth the solid is depressed.

198. If the body be irregular, and the section of the vessel be not considerable with respect to the corresponding section of the body: to determine the time of an oscillation.



Let $AEFB$ be the surface of the fluid in the vessel; and suppose that when the body is depressed through a very small space AG, that the fluid ascends in the vessel to H. Let a and b represent the sections EF and AB of the body and vessel. Let $AG = x$, and M and M' be the magnitudes of the body and the part EQF immersed.

Since the portion of the body depressed is equal in magnitude to that of the water elevated;

$$AH.(b-a) = ax;$$

$$\therefore\ AH = \frac{ax}{b-a},$$

$$\text{and } HG = x + \frac{ax}{b-a} = \frac{bx}{b-a},$$

whence $\frac{abx}{b-a}$ = the quantity of water displaced.

Let s and s' be the specific gravities of the solid and fluid, and therefore the moving force upwards against the body $= \frac{abx}{b-a} \times s'$.

$$\text{Hence the accelerating force } = \frac{s'}{Ms} \times \frac{abx}{b-a}$$

$$= \frac{1}{M'} \times \frac{abx}{b-a};$$

which varying as the distance from the point of equilibrium, the length of the pendulum $= M'.\frac{b-a}{ab} = M'.\left(\frac{1}{a} - \frac{1}{b}\right)$: whence the time is determined, as before.

199. Cor. If the section of the vessel be indefinitely great compared with that of the body, or b indefinitely great compared with a; the length $= M' \cdot \frac{1}{a}$.

200. Def. The *Equilibrium of Stability* is that in which the solid floats permanently in a given position; that is, when the body, if its position be changed, will have a tendency to regain its former position.

201. Def. The *Equilibrium of Instability* is that in which the solid, though the two centres of gravity of the fluid and solid are in the same vertical line, is easily overset; *or*, if its state be changed, the body instead of tending to regain its first position, has a tendency to revolve farther from it.

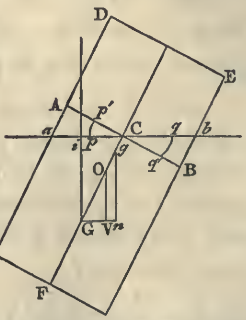
202. Def. The *Equilibrium of Indifference* is that in which the solid rests without any tendency to move from the position in which it is placed.

203. If a floating body revolve round a moveable axis, which always remains parallel to a fixed horizontal line; and in this manner be made to pass successively through all its positions of equilibrium in which the axis has the same direction; the positions of stability and instability will succeed each other alternately.

For, suppose the position of the body at first to be that of the equilibrium of stability; then whilst it continues very near to this position, it will have a tendency to return to it (200); but as the body is more reclined, and this tendency thereby diminished, it will at length endeavour to recede farther from it. Before, however, this change takes place, there will be a position in which this tendency is $= 0$; that is, where there is neither a tendency to return to, nor recede from its first position. This will be its second position of equilibrium. Now before the body arrives at this position, its tendency is to return to the first, and therefore from the second: after it has passed this, the tendency is to recede from the first, and therefore from the second. This second position therefore is one of instability, since on each side of it, the tendency is to recede from it (201).

After passing the second, the tendency to recede from it gradually decreases, till at length it becomes = 0, when it arrives at its third position of equilibrium, which is one of stability; since on each side of it the tendency is towards it. In the same manner the fourth will be one of instability, and the fifth of stability, and so on. So that when the body has returned to its original position, it will have passed through an even number of positions of equilibrium which are alternately those of stability and instability.

204. If a parallelopiped float on a fluid; to determine its stability at a small angle of inclination from a given position of equilibrium.



Let $DFBE$ be the body, G its centre of gravity, AB the plane of floatation when it is quiescent, and O the centre of gravity of the fluid displaced; therefore GO the line joining the centres is then vertical (135). Suppose now the body to be turned through a small angle θ, and ab to become the plane of floatation; the centre of gravity of the fluid displaced is no longer at O, but transferred towards those parts which have become more immersed.

Now since the weight of the whole solid remains unaltered, the magnitude of the part immersed will be the same; therefore $aFb = AFB$, and $ACa = BCb$; for if not, the weight of the body would not be equal to the pressure upwards of the fluid, and therefore these forces applied at G would produce a vertical motion, which is not here taken into the account, being considered as indefinitely small; and independent on the motion round G*.

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* If at the origin of the motion the weight of the body exceed the pressure upwards of the fluid, G will descend, and its motion be accelerated

But since $ACa = BCb$, and they are small; if we suppose them generated by the revolution of the surfaces AC, BC round the axis of floatation passing through C, and pp', qq' the paths described by their centres of gravity,

$$AC \times pp' = BC \times qq';$$

whence the products of AC and BC into their respective distances from the axis of floatation passing through C are equal; and therefore the centres of gravity of the surfaces AB, ab are situated in that axis.

Let $GO = a$, $AFB = aFb = S$; and since θ is small, $\sin\theta = \theta$.

Draw the vertical line Gi, and Gn perpendicular to it.

Then since the pressure upwards on aFb is equal to the weight of the fluid displaced, and that this force acts at the centre of gravity of aFb, let gu be a vertical line passing through that point.× Now $aFb = AFB + CBb - ACa$; and × and let if we suppose each of them collected in their respective

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lerated (107): but as it descends deeper, the pressure upwards will be increased, till it arrives at that point at which it is equal to the weight of the body, when the acceleration ceasing, G moves in the same direction in consequence of its acquired velocity, till the pressure upwards prevailing over the weight of the body, its motion is retarded and then destroyed. From this point, G having lost its velocity will again return towards its original position, and continue to oscillate till the resistance of the fluid has entirely destroyed its motion.

The length of these oscillations will be so much the less, as the difference between the weight of the body and that of the fluid displaced is less in comparison with the former of them. If the body has been only slightly deranged from its position of equilibrium, this difference will itself be very small, and therefore also the length of the oscillations; and these being inconsiderable may be supposed to have no influence on the stability of equilibrium of the body.

centres of gravity, the fluid being homogeneous, their weights will be proportional to the areas, and therefore*

$$aFb \times Gn = AFB \times GV + BCb \times qi - ACa \times pi,$$

$$= AFB \times GV + BCb \times pq;$$

$$\text{or} \quad Sx = Sa\theta + \tfrac{1}{2} BC^2 \times \theta \times \tfrac{2}{3} AB$$

$$= Sa\theta + \frac{AB^3}{12} . \theta;$$

$$\therefore \; x = \left(a + \frac{AB^3}{12S}\right) . \theta.$$

If the point O were below G, the perpendicular GV would fall in the opposite direction, and the equation would become

$$x = \left(-a + \frac{AB^3}{12S}\right) . \theta.$$

Hence therefore in general $x = \left\{\frac{AB^3}{12S} \pm a\right\} . \theta$; where the upper sign is to be used when the centre of gravity of the body is lower than that of the fluid displaced, which is the case of loaded ships; and the lower sign when the contrary.

If then W = the weight of the body or of the fluid displaced, $W . \left\{\frac{AB^3}{12S} \pm a\right\} . \theta$ = the effort made by the fluid to retain the body in its position of equilibrium, or to carry it farther from it, and will therefore be *a measure of the stability*.

205. Cor. 1. When the centre of gravity of the body is lower than that of the fluid displaced, the stability is positive, as also in the other case whilst a is less than $\frac{AB^3}{12S}$. The pres-

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* Wood's Mechanics (172). Whewell's Mechanics (62).

sure of the fluid tends to cause the body to return to its first position, or the equilibrium is that of *stability.*

If $x=0$, the centres of gravity of the body and fluid displaced are in the same vertical line, and the whole pressure produces no effect in turning the body, or the equilibrium is that of *indifference.*

If a be greater than $\frac{AB^3}{12S}$, and the centre of gravity of the body be above that of the fluid displaced, x is negative, and the centre of gravity of aFb is on the other side of Gi, and tends to cause the body to move farther from its first position; or the equilibrium is that of *instability.*

206. Def. The *Metacentre* is the intersection of the line of support with the axis passing through the centre of gravity, about which the floating body revolves through a small angle.

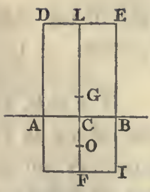
207. Cor. 2. Since $Gn = \sin\theta \times Gg = \theta . Gg$;

$$\therefore \; Gg = \frac{AB^3}{12S} \pm a;$$

and g is the metacentre; the stability therefore will be positive or negative, or nothing, according as the metacentre is above, below, or coincident with the centre of gravity of the floating body.

208. Cor. 3. In determining the positions which bodies assume on the surface of a fluid, and their stability of floating, it is necessary only to find the distance between the two vertical lines which pass through the centres of gravity of the solid and of the part immersed.

Ex. 1. To determine the stability of a homogeneous rectangular parallelopiped, floating perpendicularly to the surface of the fluid.



Let *AB* be the line of floatation of a vertical section passing through the centre of gravity *G*. Through *G* draw the vertical line *FGL*. Let $a =$ the breadth of the section of the parallelopiped, and $c =$ its length, and $n : 1$ the ratio of the specific gravities of the solid and fluid. Let *O* be the centre of gravity of the part immersed.

$$\text{Now } n : 1 :: FC : FL;$$

$$\therefore FC = nc, \text{ and } GO = \tfrac{1}{2}.(c - nc),$$

$$\text{and } AI = acn;$$

whence $x = \theta . \left\{ \dfrac{a^3}{12acn} - \tfrac{1}{2}.(c - nc) \right\}$ which varies as the stability.

Cor. 1. To determine the limits of stability and instability depending upon the dimensions and specific gravity of the solid, let $x = 0$;

$$\therefore \frac{a^3}{12acn} = \tfrac{1}{2}(c - nc),$$

$$\text{and } n^2 - n = -\frac{a^2}{6c^2};$$

$$\text{whence } n = \tfrac{1}{2} \pm \sqrt{\tfrac{1}{4} - \frac{a^2}{6c^2}}.$$

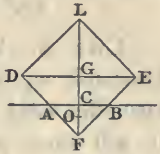
Cor. 2. When $\dfrac{a^2}{6c^2}$ is less than $\frac{1}{4}$, or when the height of the solid has a greater proportion to the base of the section than $\sqrt{2} : \sqrt{3}$, two values may be assigned to the specific gravity of the body which will cause it to float in the equilibrium of indifference.

If, for instance, $c = a$, $n = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{1}{6}} = 0.78868$ and 0.21132.

Cor. 3. If the specific gravity of the body be very small compared with that of the fluid, $\frac{a^3}{12acn}$ must be greater than $\frac{1}{2}(c-nc)$ and the solid will float permanently with DE parallel to the horizon.

Cor. 4. If the ratio of the height to the base of the section be less than $\sqrt{2}:\sqrt{3}$, no value can be given to the specific gravity which will cause the stability to vanish; because $\sqrt{\frac{1}{4}-\frac{a^2}{6c^2}}$ becomes impossible: in which case the solid placed with the surface DE horizontal must in all cases continue to float permanently in that position, whatever may be the specific gravity, supposing it always less than that of the fluid.

Ex. 2. To determine the stability of a square parallelopiped, when one of the diagonals of a vertical section is in a vertical position.



Let $LDFE$ be a vertical section, AB the line of floatation, G and O the centres of gravity of the solid and part immersed, and $n:1$ the ratio of the specific gravities of the solid and fluid.

$$\text{Let } EF=a\text{; } \therefore\ GF=\frac{a}{\sqrt{2}}.$$

$$\text{And since } CB=CF\text{; } \therefore\ AFB=CB^2.$$

$$\text{Now } ABF : DLFE :: n : 1\text{;}$$

$$\therefore\ CB^2=AFB=n\,.\,DLFE=na^2,$$

$$\text{and } CB=\sqrt{n}\,.\,a,$$

$$\text{whence } AB=2\sqrt{n}\,.\,a,\quad OF=\tfrac{2}{3}\,.\,\sqrt{n}\,.\,a,$$

$$\text{and } GO=\frac{a}{\sqrt{2}}-\frac{2a\sqrt{n}}{3}=\frac{a}{3\sqrt{2}}\,.\,\{3-2\sqrt{2n}\}\text{;}$$

$$\therefore\ x=\theta\,.\left(\frac{AB^3}{12S}-GO\right)=\theta\,.\left\{\frac{8a^3n^{\frac{3}{2}}}{12a^2n}-\frac{a}{3\sqrt{2}}\,.(3-2\sqrt{2n})\right\}.$$

Cor. 1. In order to obtain the limit separating the cases of stability and instability, $x=0$, or

$$\frac{8a^3n^{\frac{3}{2}}}{12a^2n}=\frac{a}{3\sqrt{2}}\cdot(3-2\sqrt{2n});$$

$$\therefore\ 2n^{\frac{1}{2}}=\frac{1}{\sqrt{2}}\cdot(3-2\sqrt{2n}),$$

$$\text{and}\ \ 4\sqrt{2n}=3;$$

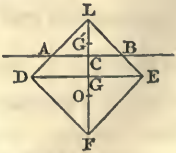
$$\therefore\ n=\tfrac{9}{32}=0.28125,$$

the specific gravity which will cause the solid to float in the insensible equilibrium; and therefore is the limit separating the specific gravities which cause the solid to float with stability from those which produce the equilibrium of instability.

Cor. 2. When n is evanescent, the solid will overset when placed on the fluid with an angle upwards, $\frac{8a^3n^{\frac{3}{2}}}{12a^2n}$ being less than $\frac{a}{3\sqrt{2}}\cdot(3-2\sqrt{2n})$.

Cor. 3. When $n:1::9:32$, the solid floats in the insensible equilibrium; if the specific gravities are in a less ratio, the solid will overset; but if they are in a greater ratio, it will float permanently with LF vertical, or the angle upwards.

Ex. 3. To determine the stability of a square parallelopiped with one of its angles upwards, when its specific gravity is greater than half the specific gravity of the fluid.



Let $LDEF$ be a vertical section in this case; and retaining the notation of the last Example; $ABEFDA=na^2$,

$$\text{and}\ \ LC^2=LAB=a^2-na^2;$$

$$\therefore\ LC=a\sqrt{1-n},\ \ AB=2a\sqrt{1-n},$$

$$\text{and}\ \ GC=a\cdot\left\{\frac{1}{\sqrt{2}}-\sqrt{1-n}\right\}.$$

Let G' be the centre of gravity of LAB; then
$GC \times$ area $DLEF =$ area $ABEFDA \times OC - ALB \times CG'$,

$$\text{or } a^3 . \left\{\frac{1}{\sqrt{2}} - \sqrt{1-n}\right\} = a^2 n . OC - \frac{a^3}{3} . (1-n)^{\frac{3}{2}};$$

$$\therefore \; OC = \frac{3a - 3\sqrt{2} . \sqrt{1-n} . a + \sqrt{2} . (1-n)^{\frac{3}{2}} . a}{3\sqrt{2} . n}$$

$$= \frac{a}{3\sqrt{2} . n} . \{3 . (1 - \sqrt{2} . \sqrt{1-n}) + \sqrt{2} . (1-n)^{\frac{3}{2}}\};$$

$$\therefore GO = OC - CG = OC - \frac{a}{3\sqrt{2} . n} . \{3n - 3n\sqrt{2} . \sqrt{1-n}\}$$

$$= \frac{a}{3\sqrt{2} . n} . \{3 - 3n - 3\sqrt{2} . (1-n)^{\frac{3}{2}} + \sqrt{2} . (1-n)^{\frac{3}{2}}\};$$

$$\therefore \; x = \theta . \left(\frac{AB^3}{12S} - GO\right)$$

$$= \theta . \left\{\frac{8a^3 . (1-n)^{\frac{3}{2}}}{12a^2 n} - \frac{a}{3\sqrt{2} . n} . [3 - 3n - 2\sqrt{2} . (1-n)^{\frac{3}{2}}]\right\} .$$

Cor. In order to obtain the limit, $x = 0$;

$$\therefore \; 2 . (1-n)^{\frac{3}{2}} = \frac{1}{\sqrt{2}} . \{3 . (1-n) - 2\sqrt{2} . (1-n)^{\frac{3}{2}}\},$$

$$\text{or } 4\sqrt{2} . \sqrt{1-n} = 3;$$

$$\therefore \; 1 - n = \frac{9}{32},$$

$$\text{and } n = \frac{23}{32}.$$

209. The fluid being considered as non-resisting, the floating body will oscillate about an horizontal axis passing through the centre of gravity: to determine the nature of the oscillations, supposing them indefinitely small.

Suppose the vertical line passing through the centre of gravity of the body when in equilibrio, to have been inclined by a quantity m. If θ be the inclination of the body at the end of the time t, and a the arc described by a point in GO at the distance 1 from G, $\theta = m - a$.

Let $v =$ the velocity of this point, or the angular velocity. At the end of the time t the pressure upwards of the fluid tends to increase this velocity; and the angular accelerating force $= \frac{dv}{dt}$; to determine which, we must divide the sum of the momenta of the moving forces by the moment of inertia; that is, if $A = \frac{AB^3}{12S} \pm a$, and $Mk^2 =$ the sum of the products of each particle × the square of its distance from the axis passing through the centre of gravity; since $W = gM$,

$$\frac{dv}{dt} = \frac{Ag\theta}{k^2} = \frac{Ag}{k^2} \times (m - a);$$

$$\text{but } v = \frac{da}{dt};$$

$$\therefore\ v\,dv = \frac{Ag}{k^2} \times (m - a) \, .\, da,$$

$$\text{and } v^2 = \frac{Ag}{k^2} \, . \, (2ma - a^2) + C;$$

$$\text{but when } v = 0,\ a = 0;\ \therefore\ C = 0;$$

$$\text{and } \therefore\ v = \sqrt{\frac{Ag}{k^2}} \, . \, \sqrt{2ma - a^2};$$

$$\text{whence } dt = \frac{k}{\sqrt{Ag}} \, . \, \frac{da}{\sqrt{2ma - a^2}},$$

$$\text{and } t = \frac{k}{\sqrt{Ag}} \times \text{circ. arc, } \left[\cos = \frac{m - a}{m}\right] + C'.$$

$$\text{But when } t = 0,\ a = 0;\ \text{and } \therefore\ C' = 0;$$

$$\text{whence } \frac{Ag\sqrt{t}}{k} = \text{circ. arc, } \left[\cos = \frac{m - a}{m}\right],$$

$$\text{and } \frac{m-a}{m} = \cos\frac{Ag\sqrt{t}}{k};$$

$$\therefore\ a = m\,.\left\{1-\cos\frac{Ag\sqrt{t}}{k}\right\}.$$

If now $\frac{t\sqrt{Ag}}{k} = 0,\ \frac{1}{2}\pi,\ \pi,\ \frac{3}{2}\pi,\ 2\pi$, respectively; the values of a are $0,\ m,\ 2m,\ m,\ 0$; therefore the body after describing an arc m, will ascend on the other side through an arc m in the same time.

$$\text{Also when } \frac{t\sqrt{Ag}}{k} = \pi,\ a = 2m,$$

and the time of a complete oscillation $= \frac{\pi k}{\sqrt{Ag}}$.

And since $A = \frac{b^3}{12S} + a$,

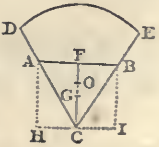
$$\frac{\pi k}{\sqrt{Ag}} = \frac{\pi k}{\sqrt{g}} \times \sqrt{\frac{12S}{12Sa+b^3}}$$ which does not involve m, and therefore the oscillations will be isochronous.

210. Cor. Since in general, $T = \pi\sqrt{\frac{L}{g}}$,

the length of the isochronous pendulum $= \frac{k^2}{A} = \frac{k^2 . 12S}{12Sa+b^3}$.

Ex. 1. Suppose the body a prism, whose transverse section is DCE, so that ABC the part immersed may be an isosceles triangle.

Let $AB = b = 2r = 2AF$,
$CF = h$.



$ADEB$ may be of any figure. Let G and O be the centres of gravity of the body and the part immersed; $\therefore FO = \frac{1}{3}FC$. Let $FG = n$;

$$\therefore\ OG = n - \tfrac{1}{3}h, \text{ and } ACB = rh;$$

$\therefore$ the distance of the metacentre from $G = \dfrac{2r^2 \pm h.(3n-h)}{3h}$, which determines the conditions necessary for the body's being in stable, unstable, or permanent equilibrium.

And the length of the isochronous pendulum

$$= \frac{3hk^2}{2r^2 \pm h.(3n-h)}.$$

Ex. 2. If the part immersed $ABIH$ be a rectangle,

let $AB = b = 2r$, $BI = h$, $GF = n$,

$S = 2rh$, $OF = \frac{1}{2}h$, and $GO = n - \frac{1}{2}h$,

and the distance of the metacentre from $G = \dfrac{r^2}{3h} \pm (n - \frac{1}{2}h)$.

And the length of the isochronous pendulum

$$= \frac{3hk^2}{r^2 \pm 3h.(n - \frac{1}{2}h)}.$$

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SECT. IV.

211. THE velocity of a stream at any part of a pipe or open channel, varies inversely as the area of the section of the pipe or channel, perpendicular to the axis at that part; supposing the pipe to continue always full, or the depth of the fluid in the open channel, to be always the same at the same place.

For the same quantity of fluid which passes through one section, will in the same time pass through the next; else, either the fluid between the sections will be condensed, which is contrary to the nature of an incompressible fluid; or there will be a vacuity in the stream, which is contrary to the supposition. Hence, if A and A' be the areas of the sections, and V and V' the velocities of the particles at those sections,

$$A \times V = A' \times V';$$

$$\therefore\ A : A' :: \frac{1}{V} : \frac{1}{V'},$$

$$\text{or } A \propto \frac{1}{V}.$$

The changes in the diameters of the sections are here supposed to be *continual.* If there be any angles in the pipe, they will produce eddies in the motion of the fluid, and the proposition will not hold true.

212. COR. 1. If the areas of the two sections are as $1 : \sqrt{2}$, the spaces due to the velocities at the sections, are as $2 : 1$*.

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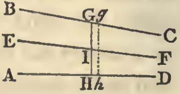
* Wood's Mechanics, 241. Whewell's Mechanics, 178.

213. Cor. 2. The same demonstration is applicable to different sections of a fluid issuing through the orifice of a vessel, whether the section be taken within or without the vessel; provided the fluid between the sections suffer no vacuity.

214. Cor. 3. If the area of the orifice be indefinitely small, compared with the area of the base of the laminæ, into which the fluid may be supposed to be divided; the mean velocity of the fluid at the orifice will be indefinitely greater than that of the laminæ, that is, while the velocity at the orifice is finite, that of the laminæ will be indefinitely small.

215. If water flows in a channel, in a direction inclined to the vertical at an angle ϕ; to determine the velocity at any section, the resistance not being considered.



Let $ABCD$ be a section of the channel, whose base is AD and surface BC; the water flowing in the direction EF, which makes with the vertical the angle ϕ. Let $v =$ the velocity of the section GH corresponding to the abscissa $AH = x$. Draw gh parallel and indefinitely near to GH.

The gravity (g) which accelerates the lamina $GHhg$ may be resolved into two, one in the direction EF, and the other perpendicular to it. The former of these will $= g . \cos. \phi$; and by the supposition, the pressure on GH is equal to that on gh in the opposite direction, whence the only accelerating force $= g . \cos. \phi$;

$$\therefore\ g dx . \cos. \phi = v dv,$$

from which equation the velocity is determined.

216. Cor. If s and S be the spaces respectively due to the velocities of the section GH, and the origin of the channel AB,

$$s = S + x . \cos \phi.$$

For $x . \cos \phi + C = \dfrac{v^2}{2g} = s$,

$$\text{and when } x = 0, \quad s = S;$$

$$\therefore\ s = S + x \cdot \cos\phi.$$

217. If the section of the channel be rectangular, and of uniform width; to determine the equation of the curve *BGC*, the surface of the water.

Let $h = AB$, the height of the first section, and $y =$ the height of the section *HG*. The altitudes s, S due to the velocities are reciprocally proportional to the squares of the sections (211), or of their altitudes y, h;

$$\therefore\ s = S \cdot \frac{h^2}{y^2},$$

$$\text{and } S \cdot \frac{h^2}{y^2} = S + x \cdot \cos\phi;$$

$$\therefore\ S \cdot \left(\frac{h^2}{y^2} - 1\right) = x \cdot \cos\phi, \text{ the equation required.}$$

218. Cor. 1. The curve is of the hyperbolic kind; the convexity being towards the bottom of the channel, and converging to it as an asymptote. When the channel has only a small declivity, the curvature of the surface is inconsiderable, and may be considered as a right line parallel to the bottom. And if the bottom be horizontal, the surface becomes a horizontal plane, and the water flows uniformly.

219. Cor. 2. If the width of the channel varied, or the sections were not rectangular, the curve would be different. When, however, the width is constant, or considerably exceeds the height of the sections, as frequently happens in rivers, the curvature of the surface will differ but little from what has been determined.

220. If the resistance to the flowing of water in pipes, or the channels of rivers be uniform, and equal to gR; to determine the velocity.

In this case the accelerating force $= g \cdot \cos\phi - gR$;

$$\therefore\ v\,dv = g\,dx \cdot \cos\phi - gR \cdot dx,$$

from which equation v may be determined.

221. Cor. In this case the curve BGC will be determined from the equation

$$\frac{dy}{dx} = \frac{y^3}{2h^2S} \cdot (R - \cos\phi).$$

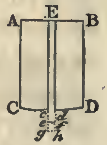
For $v = \frac{h}{y} \cdot \sqrt{2gS}$ ......(211);

$$\therefore\ v\,dv = -\frac{h^2dy}{y^3} \cdot 2gS,$$

$$\text{and } g\,dx \cdot \cos\phi - gR\,dx = -\frac{h^2dy}{y^3} \cdot 2gS;$$

$$\therefore\ \frac{dy}{dx} = \frac{y^3}{2h^2S} \cdot (R - \cos\phi).$$

222. The velocity of a fluid issuing from an indefinitely small orifice in the bottom or side of a vessel, kept constantly full, is equal to that which is due to the depth of the orifice.



Let AB be the surface of the fluid, and cd the indefinitely small orifice. Suppose the fluid to be divided into an indefinite number of horizontal laminæ, which during their descent continue parallel. Since the area of the orifice is indefinitely small compared with the sections of the fluid in the vessel (214), the velocity of the descending laminæ will be evanescent; and that which would be generated by gravity in the descending particles will be lost. The small column $cdhg$, therefore, which is discharged every instant, must be expelled by the superincumbent column Ecd, or by a moving force $s \times Ec \times O$, if s be the specific gravity of the fluid, and O the area of the orifice.

Let now $cdfe$ be a column which would have been discharged by gravity alone in the same time, and let V and V' be the velocities generated in the columns $cdhg$, $cdfe$ by the moving forces $s \times Ec \times O$, $s \times ec \times O$; these moving forces will be proportional to the quantities of motion; and

$$s \times Ec \times O : s \times ec \times O :: V \times cdhg : V' \times cdfe,$$

$$\text{or } Ec : ec :: V \times cd^2 \times cg : V' \times cd^2 \times ce$$
$$:: V \times cg : V' \times ce$$
$$:: V^2 : V'^2,$$

since $cg : ce :: V : V'$, being described in equal times.

Now if $v =$ the velocity due to the height Ec,

$$v^2 : V'^2 :: Ec : ec :: V^2 : V'^2;$$
$$\therefore\ v = V,$$

and the velocity with which the fluid issues from cd is equal to that due to the depth Ec.

223. Cor. 1. If $h =$ the depth of the orifice, and $g =$ the force of gravity; the velocity with which water issues $= \sqrt{2gh}$. Hence, if with a parameter $= 2g$, and axis the perpendicular side of a vessel, the vertex being at the upper surface of the fluid, a parabola be described, the velocity with which the fluid would issue from small orifices in the side, would be represented by the corresponding ordinates.

224. Cor. 2 If any pressure be exerted on the surface of the fluid, the velocity of the issuing fluid will be increased; and this increase will be such as would arise from supposing the upper surface raised till the pressures become equal.

Thus when water is projected into a vacuum, as the pressure of the atmosphere is equal to that of a column of water, whose altitude is 34 feet, $v = \sqrt{2g \cdot (h + 34)}$. And in general, if h' be the height of the column of fluid which would exert the same pressure as is applied at the upper surface,

$$v = \sqrt{2g \cdot (h + h')}.$$

225. Cor. 3. It is evident also that pressure might be substituted for the weight of the incumbent column of fluid. And the jet thus produced will have the velocity due to the altitude of a column of fluid which would produce that pressure.

226. The accuracy of the general conclusion (222), is affected by several circumstances. It supposes the orifice to be suddenly opened, and determines only the velocity of the first effluent water. The whole mass of fluid is, however, put in motion; and the particles approaching towards the orifice are not expelled by the same pressure. Their mutual attraction also prevents their easy separation and escape: and their tendency towards the centre of the orifice, produces a contraction of the issuing stream. To ascertain the effects produced by these causes, we must have recourse to actual experiments.

Exp. 1. If during the efflux of water from vertical prismatic vessels, whose orifices are small, minute particles be thrown in, whose specific gravity is a little greater than that of water; they will descend vertically till they reach a distance from the orifice, equal to three radii of the orifice, when they incline on every side towards the orifice, describing curves which are convex towards the axis of the vessel. So that the stream of water near the orifice forms a converging conoid, the altitude of which is three radii of the orifice, the upper base being the section of the vessel, and the lower the area of the orifice.

The small portion of water which surrounds the conoid remains stagnant near the edges of the vessel.

This concourse of all the particles to the orifice, through a conoidal funnel, equally obtains whether the orifice be in the base, or side of the vessel.

Exp. 2. If on the water there be placed a stratum of oil, or any coloured fluid lighter than water, this fluid when it has descended to the distance of three radii of the orifice, will pass through the water to reach the orifice. This experiment shews the conoid more distinctly to be converging and convex towards the axis.

Exp. 3. When the surface in which the orifice is made is thin, the issuing stream continues to contract externally through a small space, in the same oblique and converging direction; so that externally another conoid is formed, which may be considered as a continuation of the former. This is called the *vena contracta:* the lowest section of which, or the section at the greatest contraction is called the *section of the vena contracta.*

Exp. 4. The section of the vena contracta is distant from the orifice a little less than the radius of the orifice; and its magnitude is about $\frac{5}{8}$ of the magnitude of the orifice*.

Exp. 5. The situation and measure of the contraction remains the same, though the direction of the jet, or the height of the vessel vary; the orifice being small in comparison of the section of the vessel.

227. Cor. 1. The velocity of the water descending in the vessel being almost insensible, and that of efflux finite, the acceleration takes place in the space comprised within the conoids. In this space as the sections of the stream decrease rapidly, so the velocity increases rapidly.

This prismatic tube therefore should be considered as terminated by a converging tube formed of these conoids. The exact form of this additional tube is not known, but its length is about four radii of the orifice.

228. Cor. 2. In applying the theory therefore instead of the area of the orifice, we must substitute that of the

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* The ratio is not constant. It will undergo variations by varying the form of the orifice, the thickness of the surface in which the orifice is made, the form of the vessel, &c. From different experiments the values of 0.597, and 0.620 have been deduced. But in practice where extreme nicety is not required 0.625 or $\frac{5}{8}$ may be retained.

If a cylindrical tube, whose length is rather more than twice its diameter, be fixed at the orifice, the quantity discharged is $\frac{13}{16}$ of the quantity which theory would assign.

section of the vena contracta, and reckon for the height of the vessel, the height of the surface above the centre of that section. Indeed the mutual action of the strata, and the acceleration which is the effect of it, do not terminate at the orifice, but continue to the section of the vena contracta; which must therefore be regarded as the lowest section.

229. If a fluid issues through an oblique pipe in the side of a vessel which is kept full, the directrix of the parabola described is the same, whatever be the angle of elevation.

For the fluid issues with the same velocity, whatever be the direction of the pipe.

230. When a fluid spouts directly upwards, it rises nearly to the height of the stagnant fluid.

For the velocity of the issuing fluid, is that which is due to the depth of the orifice (222), very nearly. And since a body ascends through the same space, to lose any velocity, through which it would fall, to acquire it by the same force, the fluid will rise nearly to the altitude of the stagnant fluid. This is confirmed by experiments *.

231. Cor. If the density of the compressing be different from that of the projected fluid, the altitude of the jet will also vary. Thus a column of mercury acting on water, will expel it with a velocity due to an altitude repeated 14 times nearly. And the action of a cylinder of water on mercury would only raise it to $\frac{1}{14}$th part of its own height.

232. The causes which prevent a fluid from rising to the altitude of the stagnant fluid are friction, tenacity, the resistance of the air, and the re-action of those particles of fluid which have already attained their greatest altitude.

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* Those succeed best, wherein the altitude is not great; because then the velocity and space passed over being less, the effect produced by the air's resistance is less.

233. Cor. Hence, a *small* inclination of the jet causes the water to rise higher than when it is projected vertically. For the descending fluid falling a little to one side, no longer opposes the ascent of the rising fluid.

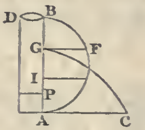
234. The height and distance to which a fluid spouts, and the time of rising and falling at different inclinations of the jet may be determined.

For every section of the stream perpendicular to its axis, may be considered as a projectile; and therefore its range, greatest altitude, and time of motion, may be determined by the rules applied to other projectiles.

235. If a cylinder or prism be placed upright on a horizontal plane, the range of a fluid issuing through a very small orifice in its side, will be equal to twice the sine of a circular arc, whose diameter is equal to the depth of the fluid, and versed sine the depth of the orifice.

The orifice being small, its radius is also; and therefore the vertex of the parabola described may be supposed to coincide with it.

Let then AB be the altitude of the fluid, G the orifice, GC the parabola described, and AC the horizontal range of the fluid. On AB describe a semi-circle, and through G draw GF perpendicular to AB.



The velocity at G is equal to that acquired down BG (222); and therefore the parameter to the point $G = 4BG$.

$$\text{Hence, } AC^2 = 4BG \times GA = 4GF^2,$$

$$\text{and } AC = \cancel{EGF}, \quad 2\,GF$$

or the horizontal range $= 2 \,.\, \sin BF$.

236. Cor. 1. Hence, the horizontal range $\propto \sin\theta$; where $\theta =$ the arc of a circle, whose diameter is the depth of the fluid, and versed sine the depth of the orifice.

237. Cor. 2. The time in which a section of the fluid moves from the orifice to the plane is equal to the time in which a body descends freely down a space $= GA$.

238. Cor. 3. When the orifice is at the point bisecting the altitude of the fluid in the vessel, the fluid will spout to the greatest distance on the horizontal plane AC; and that distance is equal to the depth of the fluid.

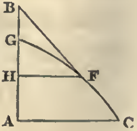
$$\text{For } AC = 2 \,.\, \sin\theta = 2 \text{ rad.} = AB.$$

239. Cor. 4. If an orifice be made at a point I below the middle, at a distance from it equal to the distance of G, the issuing fluid will strike the plane at C.

240. Cor. 5. Since the distance to which the fluid spouts, depends solely upon the height of its surface DB above the orifice; the horizontal range AC will be the same whether the fluid reaches down to A, or the bottom of the vessel stands at some higher point P; and will be the greatest when the fluid issues from an orifice half way between the planes DB and AC. If the bottom of the vessel is higher than the middle point, the nearer to that point the orifice is placed, the greater will be the distance on AC to which the fluid will spout.

241. Cor. 6. If the side of the vessel be inclined to the horizon at any angle, and the velocity and direction of the fluid be given, the range may be found.

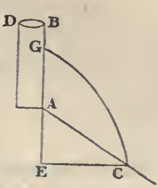
242. Cor. 7. If GH be made $= GB$, and the ordinate HF drawn, and BF joined; BF will touch the parabola.



$$\text{Now } HF^2 = 4BG \times GH = 4GH^2;$$
$$\therefore\ HF = 2GH = HB,$$
$$\text{and the angle } HBF = 45^\circ;$$

hence if orifices be made in every point in BA, all the issuing streams will be touched by a straight line drawn from B, and forming an angle of 45° with BA.

Prob. If an erect cylinder stand on the top of an inclined plane making a given angle with the horizon; to determine where a small orifice must be made, so that the fluid may strike the plane at the greatest distance.



Suppose the fluid issuing from G to strike the plane at C. Draw the horizontal line CE meeting BA produced in E.

When the range is a max., AE as also EC is a max. But (240) EC is a max. when the orifice G is at the point of bisection of BE, and $\therefore EC = EB$ or $2BG$ (238).

$$\text{But } EC \times \tan ACE = AE,$$

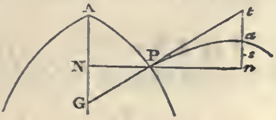
$$\text{or } 2BG \times \tan ACE = 2BG - BA;$$

$$\therefore BG = \frac{BA}{2.(1-\tan ACE)}.$$

Cor. If $ACE = 30^0$, $\tan 30^0 = \frac{1}{\sqrt{3}}$;

$$\therefore BG = BA.\frac{\sqrt{3}}{2.(\sqrt{3}-1)} = BA.\frac{3+\sqrt{3}}{4}.$$

Prob. A paraboloid resting on its base is kept constantly filled with fluid. Find at what point a very small orifice must be made, that the latus rectum of the parabola described by the issuing fluid may be half the latus rectum of the vessel.



Suppose P to be the orifice, Pa the parabola described by the fluid.

Let $AN = x$, $an = x'$,
$PN = y$, $Pn = y'$,

$4a =$ the latus rectum of AP;

$\therefore 2a =$ the latus rectum of Pa.

$$\begin{aligned} \text{Then} \quad Pn &: nt :: PN : NG, \\ \text{or} \quad y' &: 2x' :: y : 2a; \\ \therefore x'^2 &: a^2 :: y'^2 : y^2 \\ &:: 2ax' : 4ax \\ &:: x' : 2x; \\ \therefore x' &= \frac{a^2}{2x}. \end{aligned}$$

Now the space due to the velocity at $P = \frac{1}{4}$ of the parameter at the point P in the parabola Pa, $=$ the distance of P from the focus $= as + an = x' + \frac{1}{2}a$;

$$\therefore (222) \; x' + \tfrac{1}{2}a = x,$$

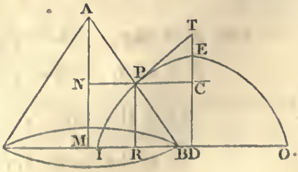
$$\text{and} \; x' = x - \tfrac{1}{2}a,$$

$$\text{whence} \; x - \tfrac{1}{2}a = \frac{a^2}{2x},$$

and $\therefore x = a$, and N is the focus.

Cor. Since $x' = x - \frac{1}{2}a = \frac{1}{2}a = \frac{1}{4}$ latus rectum of Pa; therefore n is the focus of Pa; or the foci of both parabolas are in the straight line NPn.

Prob. A hollow cone, whose vertical angle is 60°, is filled with water and placed with its base downwards. It is required to determine the place where a small orifice must be made in its side, so that the issuing fluid may strike the horizontal plane in a point whose distance from the bottom of the vessel is to the distance of the orifice from the top :: 5 : 4.



Let $\left.\begin{aligned} AN &= x \\ AM &= a \end{aligned}\right\}$; $\therefore NM = a - x$,

and $AP = \dfrac{2x}{\sqrt{3}}$.

Also by the supposition,

$$BO : AP :: 5 : 4;$$

$$\therefore\ BO = \frac{5x}{2\sqrt{3}}.$$

$$\text{And}\quad BR = PR \, . \tan 30^\circ = \frac{a-x}{\sqrt{3}};$$

$$\therefore\ RO = \frac{5x}{2\sqrt{3}} + \frac{a-x}{\sqrt{3}} = \frac{3x+2a}{2\sqrt{3}}.$$

And since * by the theory of projectiles,

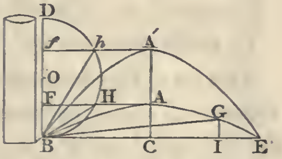
$$u = z \, . \tan\theta - \frac{z^2}{4H \, . \cos^2\theta},$$

$$x - a = \frac{3x+2a}{2\sqrt{3}} \, . \tan 30^\circ - \left(\frac{3x+2a}{2\sqrt{3}}\right)^2 \times \frac{1}{4x \, . \cos^2 30^\circ}$$

$$= \frac{3x+2a}{6} - \frac{(3x+2a)^2}{36x},$$

whence by the solution of a quadratic, $x = \frac{2a}{3} \, . \left(1 \pm \sqrt{\frac{2}{3}}\right)$.

243. To determine the horizontal range of a fluid issuing through an oblique jet.



Let B be the orifice which is very small, and BH the direction of the issuing fluid. On BD describe a semi-circle meeting BH in H, and let fall the perpendicular HF, and make $HA = HF$. From A draw AC perpendicular to BE the horizontal plane, and with the vertex A, and parameter $= 4BD$ (AC being the axis) describe a parabola BAE. BE will be the horizontal range.

For $BE = 2BC = 4FH = 4BO \times \sin 2\theta = 2H \times \sin 2\theta$*, where H is the height due to the velocity of projection, and θ the angle of elevation.

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* Whewell's Mechanics (191).

† Wood's Mechanics (333). Whewell (189. Cor. 5.).

244. Cor. 1. If $Df = BF$, then $fh = FH$; and if a parabola $BA'E$ be described, the vertex will be in the perpendicular CA, and the parabolas will meet in E. This is also true for every other pair of parabolas whose vertices are equidistant from a horizontal line passing through the centre of the circle.

245. Cor. 2. The range is the greatest when the inclination of the jet is 45°.

246. Cor. 3. The directrix of all the parabolas described will be in the horizontal line drawn through the upper surface of the fluid: and the foci will lie in the circumference of the circle described from the centre B, with the radius BD.

247. Cor. 4. If the ground be not horizontal, but inclined at a given angle GBE; to find the range.

Since $u = z \,.\, \tan\theta - \dfrac{z^2}{4H \,.\, \cos^2\theta}$,

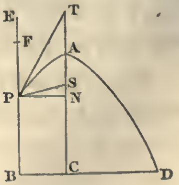
if $BI = x$ and $GBI = \alpha$,

$$x \,.\, \tan\theta - \frac{x^2}{4H \,.\, \cos^2\theta} = IG = x \tan\alpha,$$

$$\therefore\ x = 4H \,.\, \cos^2\theta \,.\, (\tan\theta - \tan\alpha);$$

$$\therefore\ \text{range} = x \,.\, \sec\alpha = 4H \,.\, \cos^2\theta \,.\, \sec\alpha \,.\, (\tan\theta - \tan\alpha).$$

Prob. The inclination of a small tube in the side of a cylindrical vessel of water being given, and its height above the horizontal plane; it is required, from observing the point of the plane struck by the stream, to assign the altitude of the water within the vessel.



Let P be the orifice, and PT the direction of the issuing fluid, inclined to the horizon at an angle α; and FP the height of the fluid above the orifice.

Let $BD = b$, $PB = c$, and $FP = H$;

$$\therefore\ -c = b \,.\, \tan\alpha - \frac{b^2}{4H \,.\, \cos^2\alpha},$$

$$\text{whence} \quad H = \frac{b^2}{2b \cdot \sin 2a + 4c \cdot \cos^2 a},$$

and the altitude of the water $= c + \dfrac{b^2}{2b \cdot \sin 2a + 4c \cdot \cos^2 a}$.

248. The velocities of fluids issuing from equal orifices, and the quantities uniformly discharged in equal times, are severally in the subduplicate ratio of the depths of the orifices.

For the velocity being that due to the depth of the orifice $\propto \sqrt{\text{depth}}$.

Again, suppose the quantity uniformly discharged in a given time to be formed into a column whose base is the orifice, the length of the column being the space uniformly described by a particle of the fluid during the passage of the column, with the velocity at the orifice. Its base being given, the content of such a column $\propto$ the length $\propto$ the space uniformly described in a given time with the velocity at the orifice $\propto$ the velocity at the orifice $\propto \sqrt{\text{depth}}$.

This is true of fluids of different specific gravities. For though a superincumbent column of one fluid may press with *n* times the force of a similar column of water, the column of that fluid expelled is *n* times as heavy as the similar column of water; and the resistance bearing the same proportion to the moving force, the velocities will be equal.

249. It is found by experiment, that the quantities of water discharged in equal times by different orifices, are nearly as the areas of the orifices, the altitude of the water being the same.

250. The velocity of a fluid issuing from an orifice in the side or base of a regular vessel is uniformly retarded.

For if $V =$ the velocity at the upper surface of the descending fluid, and $v =$ that at the orifice,

$$(211) \quad V : v :: O : A.$$

Now the vessel being regular, $O : A$ is an invariable ratio;

$$\therefore \; V \propto v \propto \sqrt{\text{depth}},$$

or $V \propto$ in the subduplicate ratio of the space to be described before the whole velocity is destroyed. But by Mechanics* $V \propto \sqrt{FS}$; and therefore in this case F is invariable, or the force which retards the descent of the surface is an uniform force.

251. Cor. 1. The descending surface is under the same circumstances with a body projected in a direction contrary to that in which an uniform force acts, and moving till its whole velocity is destroyed.

252. Cor. 2. The quantities of fluid discharged through an orifice in the bottom of a vessel, in equal times, decrease in the ratio of the numbers 1, 3, 5, 7, &c. taken in an inverted order.

253. Cor. 3. If $t =$ the time of emptying an upright prismatic vessel; the quantity discharged in t'' when the vessel is kept constantly full, will be double the quantity in the vessel.

For the space which the surface would describe with the first velocity continued uniform for t'', is twice the space which the surface does actually describe in that time. And the quantity discharged when the vessel is kept full, may be measured by what would be the descent of the surface, if it could move with its first velocity; therefore the quantity discharged in the first case is double that in the latter.

254. To explain the clepsydra.

Since the upper surface of a fluid descending in a regular vessel is uniformly retarded, and moves till its whole velocity is destroyed; suppose the motion to continue through n equal portions of time; then the spaces described in the first, second, third, &c. successive portions reckoned from the beginning of the motion are in the ratio of the quantities $2n-1$, $2n-3$, $2n-5$, &c. If therefore the surface of a cylindrical tube be divided into portions in the ratio of $2n-1$, $2n-3$, $2n-5$, &c. reckoned from the top of the tube, and then be filled with water or loose sand, which is suffered to issue through a small orifice in the bottom; at the end of the first, second, third, &c. portions of time, the upper surface will have successively descended through the first, second, third, &c. divisions; that is, the descent of the upper surface will mark the passage of time.

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* Wood, (245). Whewell, (178).

255. Any vessel may serve for a clepsydra: but that form is most commodious in which the fluid descends through equal portions of the vertical axis in equal portions of time.

To determine this form,

Let x = the altitude of the fluid;

$\therefore$ (222) the velocity at the orifice $= \sqrt{2gx}$.

Let y = the ordinate of the generating curve;

$\therefore$ $\pi y^2 =$ the area of the descending surface.

Let $v =$ its velocity;

$$\therefore (211)\ \sqrt{2gx} : v :: \pi y^2 : O,$$

$$\text{and } v = \frac{O\sqrt{2gx}}{\pi y^2}.$$

And since the surface descends uniformly, $\frac{O\sqrt{2gx}}{\pi y^2}$ will be equal to some constant quantity a, which will depend upon the whole height and the time in which the clepsydra will be emptied;

$$\therefore y^4 = \frac{2O^2 g}{\pi^2 a^2} \times x,$$

the equation to a parabola of the fourth order.

2. If the clepsydral vessel is prismatical, and one section a rectangle, one of whose sides $= y$, and the other $= p$, the

area of the surface $= py$,

$$\text{and } \frac{O\sqrt{2gx}}{py} = a;$$

$$\therefore y^2 = \frac{2O^2 g}{a^2 p^2} . x,$$

the equation to the common parabola.

3. If the section be a square or circle, the centre of which is in the axis of the clepsydra, the bounding curve will be (as above) a parabola of the fourth order.

4. If the horizontal sections be similar parabolas, and $A =$ the upper surface, and $A' =$ the surface corresponding to the depth x, and b and y be the abscissæ of those sections,

$$\frac{A'}{A} = \frac{y^2}{b^2}, \text{ by the property of similar figures};$$

$$\therefore A' = \frac{Ay^2}{b^2};$$

$$\text{whence } \frac{Ob^2\sqrt{2gx}}{Ay^2} = a,$$

$$\text{and } y^4 = \frac{2Ob^2 g}{a^2 A^2} . x.$$

The same will be true for any similar figures.

256. If water flows through a small orifice in the side or base of a vessel kept constantly full; to determine the relation between the quantity discharged, the area of the orifice, the depth, and the time.

Let $Q =$ the quantity discharged in the time t'',

$h =$ the depth,

(222) the velocity of efflux $= \sqrt{2gh}$;

and the vessel being kept constantly full, every particle issues with the same velocity;

$\therefore$ the quantity discharged in $1''$ is $= O\sqrt{2gh}$,

$$\text{and } Q = Ot\sqrt{2gh}.$$

257. Cor. 1. Knowing therefore three of these quantities, the fourth may be found.

$$t = \frac{Q}{O\sqrt{2gh}}, \quad O = \frac{Q}{t\sqrt{2gh}}, \quad h = \frac{Q^2}{2gO^2t^2}.$$

Ex. If the vessel be a vertical prism, the horizontal sections of which $= A$; the time in which a quantity equal to the content of the vessel would flow out (the vessel being kept constantly full) is $= \dfrac{hA}{O\sqrt{2gh}} = \dfrac{A}{O}\sqrt{\dfrac{h}{2g}}$.

258. Cor. 2. The quantities discharged by two orifices in the same time are as the areas of the orifices and the square roots of the depths jointly.

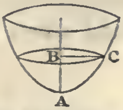
259. Cor. 3. If $n =$ the ratio of the section of the Vena contracta to that of the orifice O,

$$(228) \quad Q = nOt\sqrt{2gh},$$

$$\therefore \; n = \frac{Q}{Ot\sqrt{2gh}}.$$

If therefore an experiment would give the quantity discharged in a given time through a given orifice, and a given depth, n might be deduced, or the ratio of the sections determined.

260 To investigate an expression for the time of emptying any vessel, through an orifice which is small compared with a section of the vessel.



Let $AB = x$, $BC = y$,

$O =$ the area of the orifice,

and $S =$ that of the descending surface; which, when the figure of the vessel is known, will be given in terms of x and y.

The velocity at the orifice $= \sqrt{2gx}$;

$\therefore$ the quantity discharged in $1'' = O\sqrt{2gx}$,

$$\text{whence } O\sqrt{2gx}\,.\,dt = -S\,dx,$$

$$\text{and } dt = \frac{-S\,dx}{O\sqrt{2gx}};$$

$$\therefore\ t = C - \int \frac{S\,dx}{O\sqrt{2gx}}.$$

261. Cor. 1. If the vessel be generated by the revolution of a curve round a vertical axis, $S = \pi y^2$,

$$\text{and } t = C - \int \frac{\pi y^2\,dx}{O\sqrt{2gx}}.$$

262. Cor. 2. If any pressure be exerted on the surface of the fluid, and h' = the height of a column of the fluid which would exert the same pressure,

$$dt = \frac{-S\,dx}{O\sqrt{2g\,.\,(x+h')}}.$$

263. To find the time of emptying a cylinder or prism.

In this case S is constant, and $dt = \dfrac{S}{O\sqrt{2g}} \times \dfrac{-dx}{\sqrt{x}}$;

$$\therefore\ t = \frac{S}{O\sqrt{2g}} \times -2x^{\frac{1}{2}} + C,$$

and this vanishes when $x = h$;

$$\therefore\ 0 = \frac{S}{O\sqrt{2g}} \times -2h^{\frac{1}{2}} + C,$$

$$\text{whence } t = \frac{2S}{O\sqrt{2g}} \times (\sqrt{h} - \sqrt{x});$$

and when $x = 0$, the whole time of emptying

$$= \frac{2S\sqrt{h}}{O\sqrt{2g}} = \frac{S\sqrt{2h}}{O\sqrt{g}}.$$

264. Cor. 1. The times of emptying cylinders or prisms through equal orifices in their bases $\propto S\sqrt{h}$. If the bases be equal, the times $\propto \sqrt{h}$. And the altitudes being equal, the times are as the bases.

265. Cor. 2. The time of emptying the altitude $(h-x)$: the time of emptying the altitude $(h'-x')$

$$:: \frac{S}{O}(\sqrt{h}-\sqrt{x}) : \frac{S'}{O'}.\sqrt{h'}-\sqrt{x'}.$$

Hence the times in which the surfaces of two cylindrical vessels filled with fluid descend through any heights are in the compound ratio of their bases and the difference between the square roots of the altitudes of each surface at the beginning and end of its motion directly, and the areas of the orifices inversely.

266. Cor. 3. Since $t = \frac{2S}{O\sqrt{2g}}.(\sqrt{h}-\sqrt{x})$;

$$\therefore \sqrt{h}-\frac{O\sqrt{2g}}{2S}.t = \sqrt{x},$$

$$\text{and } h-\frac{O\sqrt{2gh}}{S}.t+\frac{O^2g}{2S^2}.t^2=x;$$

$$\therefore h-x=\frac{O\sqrt{2gh}}{S}.t+\tfrac{1}{2}.\frac{O^2g}{S^2}.t^2.$$

But in uniformly retarded motions $s=vt-\frac{1}{2}Ft^2$*.

And in this case, the initial velocity $=\frac{O}{S}\sqrt{2gh}$ very nearly;

$$\therefore F=\frac{gO^2}{S^2};$$

and in a vertical prismatic vessel the surface of the fluid descends with a motion uniformly retarded; the retarding force being $=\frac{gO^2}{S^2}$, as would appear from Art. 250.

267. Cor. 4. Since the time of emptying an altitude $(h-x) \propto \sqrt{h}-\sqrt{x}$; if h, h', h'', h''', &c. be the different

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* Wood's Mechanics (238). Whewell (180).

heights, and

$$\sqrt{h}-\sqrt{h'}=\sqrt{h'}-\sqrt{h''}=\sqrt{h''}-\sqrt{h'''}=\&c.$$

a clepsydra would be formed.

For the surfaces would descend through $h-h'$, $h'-h''$, $h''-h'''$, &c. in equal times.

268. Cor. 5. If the vessel were kept constantly full, the velocity at the orifice being $=\sqrt{2gh}$, the quantity discharged in the time t $\left(=\frac{2S\sqrt{h}}{O\sqrt{2g}}\right)$ would be

$$O\sqrt{2gh}\times\frac{2S\sqrt{h}}{O\sqrt{2g}}=2Sh=\text{twice the content of the vessel}$$

(as in 253).

269. Cor. 6. If a cylinder of given altitude empty itself through a given orifice in n'', the diameter of the cylinder may be found.

$$\text{For } S=\frac{\pi\delta^2}{4};$$

$$\therefore\ n=\frac{\frac{1}{4}\pi\delta^2\times2\sqrt{h}}{O\sqrt{2g}};$$

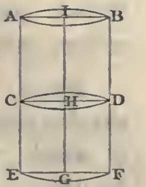
$$\therefore\ \delta^2=\frac{2nO\sqrt{2g}}{\pi\sqrt{h}};$$

$$\text{whence } \delta=\sqrt{\frac{2nO\sqrt{2g}}{\pi\sqrt{h}}}.$$

Or if the diameter be given, the height may be found $=\frac{8n^2O^2g}{\pi^2\delta^4}$.

270. Cor. 7. The time in which the surface descends to CD, or in which the quantity $ABDC$ is emptied through an orifice at $G=\frac{2S}{O\sqrt{2g}}\cdot(\sqrt{h}-\sqrt{x})$; and the time in

which the same quantity would be emptied through an orifice at $H = \frac{2S}{O\sqrt{2g}}.\sqrt{h-x}$; therefore the times of emptying the same quantity at G and H are as $\sqrt{h}-\sqrt{x} : \sqrt{h-x}$.



271. Cor. 8. The time in which AB would descend to CD with the first velocity, or the time in which a quantity equal to $ABDC$ would run out (257), supposing the vessel to be kept always full from without,

$$= \frac{S}{O\sqrt{2gh}} \times (h-x).$$

Hence the time of emptying $ABDC$ at G, when there is no supply from without, is to the time in which an equal quantity would run out, when the vessel is kept always full

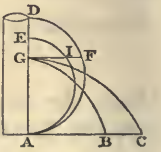
$$:: 2.(\sqrt{h}-\sqrt{x}) : \frac{h-x}{\sqrt{h}}.$$

And when $x=0$, or the surface descends to EF, the proportion becomes $2\sqrt{h} : \frac{h}{\sqrt{h}}$, or $2 : 1$.

272. It is almost impossible to ascertain the exact time in which any vessel is exhausted. For when the descending surface has nearly reached the orifice, a kind of conoidal funnel (226) is formed above the orifice; and the pressure of the superincumbent column being removed, the water falls in drops, and the time of emptying is increased. Instead therefore of endeavouring to determine the time in which vessels are completely exhausted, it is usual to determine the time in which the upper surface of the fluid descends through a certain height. And from this it appears that the times of discharge by experiment differ very little from those deduced from the corrected values of theory (228): and that the error in the latter is always in defect: which may arise from the multiplier being too great for the corrected area of the orifice.

When the orifice is in the side of the vessel, the altitude of the surface may be reckoned from the centre of gravity of the orifice, unless when it is large.

Prob. There is a hollow cylinder with a given small circular orifice at the distance of (a) feet from its bottom; out of which, when the cylinder is full, the water will spout to the distance of (b) feet from its bottom on a horizontal plane; but after it has continued running for t'', it will spout only to the distance of (c) feet. What are the dimensions of the cylinder.



Let DG and EG be the altitudes of the water above the orifice at first and after t''. Let $r =$ the radius of the cylinder, $AG = a$, $AB = c$, $AC = b$.

The time of emptying DG : time of emptying EG

$$:: \sqrt{DG} : \sqrt{EG}$$

$$:: GF : GI$$

$$:: b : c;$$

therefore the time of emptying DG

$$: (\text{time of emptying } DE =)\ t :: b : b - c;$$

$$\therefore \text{ the time of emptying } DG = \frac{bt}{b-c};$$

but (263) this time also

$$= \frac{S}{O\sqrt{g}} \times \sqrt{2\,DG} = \frac{\pi r^2}{O\sqrt{g}} \times \sqrt{\frac{AC^2}{2\,AG}}$$

$$= \frac{\pi r^2 b}{O\sqrt{2ga}};$$

$$\therefore \frac{bt}{b-c} = \frac{\pi r^2 b}{O\sqrt{2ga}},$$

$$\text{and } r^2 = \frac{Ot\sqrt{2ga}}{\pi.(b-c)}.$$

Also $DA = DG + GA = \dfrac{b^2}{4a} + a = \dfrac{b^2+4a^2}{4a}$.

And the content of the cylinder $= \dfrac{b^2+4a^2}{4a} \times \dfrac{Ot\sqrt{2ga}}{b-c}$.

273. To determine the time in which a cylinder will empty itself into a vacuum, its upper surface being exposed to the pressure of the atmosphere.

Let $h =$ the height of the vessel, and $h' =$ the height of a column of fluid, which is equal to the weight of the atmosphere.

$$\text{Then (262) } t = \int \frac{-S\,dx}{O\sqrt{2g}\,.\,\sqrt{x+h'}}$$

$$= -\frac{S}{O\sqrt{2g}} \times 2\,.(x+h')^{\frac{1}{2}} + C;$$

But when $x = h$, $t = 0$;

$$\therefore\ 0 = -\frac{S}{O\sqrt{2g}}\,.\,2\,.(h+h')^{\frac{1}{2}} + C,$$

$$\text{whence, } t = \frac{2S}{O\sqrt{2g}}\,.\,\{(h+h')^{\frac{1}{2}} - (x+h')^{\frac{1}{2}}\},$$

and the whole time $= \dfrac{2S}{O\sqrt{2g}}\,.\,\{(h+h')^{\frac{1}{2}} - h'^{\frac{1}{2}}\}$.

274. To find the time of emptying any part of a sphere.

Let $r =$ its radius, and $x =$ any depth from the bottom;

$$\therefore\ S = \pi\,.(2rx - x^2),$$

$$\text{and } dt = \frac{-\pi\,.(2rx - x^2)\,.\,dx}{O\sqrt{2gx}};$$

$$\therefore\ t = -\frac{\pi}{O\sqrt{2g}}\,.\left(\frac{4}{3}rx^{\frac{3}{2}} - \frac{2}{5}x^{\frac{5}{2}}\right) + C,$$

and when $t = 0$, $x = 2r$;

$$\therefore\ 0 = \frac{-\pi}{O\sqrt{2g}} \cdot \left(\frac{4}{3} \cdot (2r)^{\frac{3}{2}} - \frac{2}{5} \cdot (2r)^{\frac{5}{2}}\right) + C;$$

$$\therefore\ t = \frac{\pi}{O\sqrt{2g}} \times \left\{\frac{16\sqrt{2}}{15} \cdot r^{\frac{5}{2}} - \frac{4}{3} r x^{\frac{3}{2}} + \frac{2}{5} x^{\frac{5}{2}}\right\}.$$

275. Cor. Hence, the whole time of emptying a sphere $= \frac{16\pi r^{\frac{5}{2}}}{15\,O\sqrt{g}}$.

276. To determine the time of emptying any part of a hemisphere, through an orifice in the vertex.

Here, as before, $t = -\frac{\pi}{O\sqrt{2g}} \cdot \left\{\frac{4}{3} r x^{\frac{3}{2}} - \frac{2}{5} x^{\frac{5}{2}}\right\} + C,$

but $t = 0$, when $x = r$;

$$\therefore\ 0 = -\frac{\pi}{O\sqrt{2g}} \times \frac{14}{15} r^{\frac{5}{2}} + C,$$

whence $t = \frac{\pi}{O\sqrt{2g}} \cdot \left\{\frac{14}{15} \cdot r^{\frac{5}{2}} - \frac{4}{3} r x^{\frac{3}{2}} + \frac{2}{5} x^{\frac{5}{2}}\right\}.$

277. Cor. 1. When $x = 0$, the whole time $= \frac{14}{15} \cdot \frac{\pi r^{\frac{5}{2}}}{O\sqrt{2g}}$.

278. Cor. 2. Hence, the time of emptying the sphere : the time of emptying the lower hemisphere

$$:: 16\sqrt{2} : 14 :: 8\sqrt{2} : 7.$$

279. Cor. 3. And the times of emptying the successive hemispheres are $:: 8\sqrt{2} - 7 : 7$.

280. To determine the time of emptying any part of a hemisphere through an orifice in the base.

Here $y^2 = r^2 - x^2$, and $t = -\frac{\pi}{O\sqrt{2g}} \cdot \int \frac{(r^2 - x^2) \cdot dx}{\sqrt{x}}$

$$= -\frac{\pi}{O\sqrt{2g}} \times \left\{2r^2 x^{\frac{1}{2}} - \frac{2}{5} x^{\frac{5}{2}}\right\} + C,$$

and $x = r$, when $t = 0$,

$$\therefore\ 0 = -\frac{\pi}{O\sqrt{2g}} \cdot \frac{8}{5} r^{\frac{5}{2}} + C,$$

$$\text{whence, } t = \frac{\pi}{O\sqrt{2g}} \cdot \left\{\frac{8}{5} r^{\frac{5}{2}} - 2r^2 x^{\frac{1}{2}} + \frac{2}{5} x^{\frac{5}{2}}\right\}.$$

281. Cor. 1. The whole time of emptying a hemisphere through an orifice in the base $= \frac{8}{5} \cdot \frac{\pi r^{\frac{5}{2}}}{O\sqrt{2g}}$.

282. Cor. 2. The times of emptying a hemisphere through an orifice in the vertex, and in the base are as $\frac{14}{15} : \frac{8}{5} :: 7 : 12$.

283. Cor. 3. The time of emptying the upper hemisphere, at the vertex of the lower : the time of emptying it at the base of the upper one :: $8\sqrt{2} - 7 : 12$.

284. The times of emptying hyperboloids, and spheroids, may be determined in nearly the same manner.

Prob. The times in which two hemispheres are emptied, the one by an orifice in the vertex, and the other by an equal orifice in the base are as $3 : 5$. Determine the proportion of the radii.

Since (277) the time in which the first is emptied

$$= \frac{14}{15} \cdot \frac{\pi r^{\frac{5}{2}}}{O\sqrt{2g}},$$

and (281), the second $= \frac{8}{5} \cdot \frac{\pi r'^{\frac{5}{2}}}{O\sqrt{2g}}$;

$$\therefore\ 3 : 5 :: \frac{14}{15} \cdot r^{\frac{5}{2}} : \frac{8}{5} \cdot r'^{\frac{5}{2}},$$

whence $36 : 35 :: r^{\frac{5}{2}} : r'^{\frac{5}{2}}$,

and $(36)^{\frac{2}{5}} : (35)^{\frac{2}{5}} :: r : r'$.

285. To determine the time of emptying a paraboloid, through an orifice in the vertex.

Let r = the radius of the base, and h = the altitude,

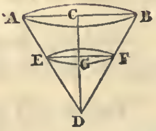
$$\text{then } S = \frac{\pi r^2 x}{h};$$

$$\text{and } t = \int \frac{-\pi r^2 x\,dx}{Oh\sqrt{2gx}} = -\frac{\pi r^2}{Oh\sqrt{2g}} \times \frac{2x^{\frac{3}{2}}}{3} + C$$

$$= \frac{2\pi r^2}{3Oh\sqrt{2g}} \times \{h^{\frac{3}{2}} - x^{\frac{3}{2}}\}.$$

And the whole time $= \dfrac{2\pi r^2 h^{\frac{1}{2}}}{3O\sqrt{2g}}$.

286. To determine the time in which a given cone will empty itself through an orifice in the vertex.



$$\text{Let } \left.\begin{aligned} DC &= a, \\ DG &= x, \\ AC &= r, \end{aligned}\right\} \text{ then } EG = \frac{rx}{a},$$

and the descending surface $= \dfrac{\pi \,.\, r^2 x^2}{a^2}$;

$$\therefore\ t = \int - \frac{\pi r^2 x^2 dx}{Oa^2\sqrt{2gx}}$$

$$= -\frac{\pi r^2}{Oa^2\sqrt{2g}} \times \frac{2}{5} x^{\frac{5}{2}} + C$$

$$= \frac{2\pi r^2}{5Oa^2\sqrt{2g}} \times \{a^{\frac{5}{2}} - x^{\frac{5}{2}}\}.$$

And the whole time of emptying $= \dfrac{2\pi r^2 a^{\frac{1}{2}}}{5O\sqrt{2g}} = \frac{1}{5}$th of the time of emptying a cylinder of the same base and altitude (263).

287. To find the time in which a cone would empty itself by an orifice in the base.

In this case, let $CG = x$,

$$\text{the descending surface} = \frac{\pi r^2}{a^2}\,.\,(a-x)^2;$$

$$\therefore\ t = \int \frac{-\pi r^2\,.\,(a-x)^2\,.\,dx}{O a^2 \sqrt{2gx}}$$

$$= -\frac{\pi r^2}{O a^2 \sqrt{2g}}\,.\int (a^2 x^{-\frac{1}{2}} dx - 2 a x^{\frac{1}{2}} dx + x^{\frac{3}{2}} dx)$$

$$= -\frac{\pi r^2}{O a^2 \sqrt{2g}}\,.\left\{2 a^2 x^{\frac{1}{2}} - \frac{4}{3} a x^{\frac{3}{2}} + \frac{2}{5} x^{\frac{5}{2}}\right\} + C$$

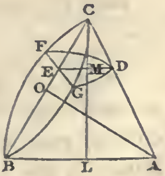
$$= \frac{\pi r^2}{O a^2 \sqrt{2g}} \times \left\{\frac{16}{15} a^{\frac{5}{2}} - 2 a^2 x^{\frac{1}{2}} + \frac{4}{3} a x^{\frac{3}{2}} - \frac{2}{5} x^{\frac{5}{2}}\right\}.$$

And the whole time of emptying $= \frac{16}{15}\,.\,\frac{\pi r^2 a^{\frac{1}{2}}}{O\sqrt{2g}} = \frac{8}{15}$ of the time of emptying a cylinder of the same base and altitude (263).

288. Cor. The times of emptying two equal cones, through equal orifices in the vertex and base, are as

$$\frac{1}{5} : \frac{8}{15} :: 3 : 8.$$

289. A cone being placed with its slant side parallel to the horizon; to determine the time of emptying it through an orifice in the vertex.



Let $AC = a$, $CB = b$, $CE = x$;

$\therefore\ DE = \frac{ax}{b}$, and $FG = 2\sqrt{x\,.\,(b-x)}$,

and $DFEG = \frac{2}{3} DE\,.\,FG = \frac{4a}{3b}\,.\,x\sqrt{x\,.\,(b-x)}$;

and $CM = x \,.\, \sin CEM = x \,.\, \sin\theta$, if $\theta = CBA$;

$$\therefore\ t = \frac{4a}{3b} \times \int x\sqrt{x\,.\,(b-x)} \times \frac{dx\,.\,\sin\theta}{O\sqrt{2g\,.\,(b-x)\,.\,\sin\theta}}$$

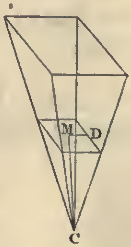
$$= \frac{4a}{3bO}\,.\sqrt{\frac{\sin\theta}{2g}} \times \frac{2}{5}x^{\frac{5}{2}} + C,$$

and when $x = 0$, $t = 0$; $\therefore\ C = 0$;

$$\therefore\ t = \frac{8a}{15bO}\,.\sqrt{\frac{\sin\theta}{2g}}\,.\,x^{\frac{5}{2}}.$$

And the whole time $= \dfrac{8ab^{\frac{3}{2}}\sqrt{\sin\theta}}{15O\sqrt{2g}}$.

290. To find the time of emptying a pyramid whose base is a square.



Let $CM = x$, $MD = y$, and the height of the pyramid $= h$; then it is evident that $x \propto y$; and therefore suppose $y = nx$; and the side of the square $= 2y$;

$\therefore$ the descending surface $= 4y^2 = 4n^2x^2$;

and $t = \displaystyle\int \frac{-4n^2x^2dx}{O\sqrt{2gx}} = -\frac{4n^2}{O\sqrt{2g}} \times \frac{2}{5}x^{\frac{5}{2}} + C$

$$= \frac{8n^2}{5O\sqrt{2g}} \times \{h^{\frac{5}{2}} - x^{\frac{5}{2}}\}.$$

And the whole time $= \dfrac{8n^2h^{\frac{5}{2}}}{5O\sqrt{2g}}$.

291. A groin is generated by a square moving parallel to itself, the section through its opposite sides being a semicubical parabola; find the time of emptying through a given orifice in the vertex.

Since $y^3 = ax^2$; $\therefore$ a side of the square $= 2y = 2a^{\frac{1}{3}}x^{\frac{2}{3}}$,

and the descending surface $= 4a^{\frac{2}{3}}x^{\frac{4}{3}}$;

$$\therefore\ t = \int - \frac{4a^{\frac{2}{3}}x^{\frac{4}{3}}dx}{O\sqrt{2gx}} = -\frac{4a^{\frac{2}{3}}}{O\sqrt{2g}} . \int x^{\frac{5}{6}}dx$$

$$= -\frac{4a^{\frac{2}{3}}}{O\sqrt{2g}} . \frac{6}{11}x^{\frac{11}{6}} + C$$

$$= \frac{24a^{\frac{2}{3}}}{11O\sqrt{2g}} . (h^{\frac{11}{6}} - x^{\frac{11}{6}}).$$

And the whole time of emptying $= \dfrac{24a^{\frac{2}{3}}h^{\frac{11}{6}}}{11O\sqrt{2g}} = \dfrac{24b^2h^{\frac{1}{2}}}{11O\sqrt{2g}}$,

if $b =$ the extreme ordinate.

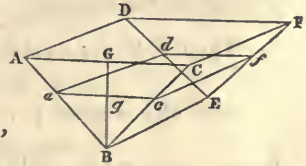
292. A cube is bisected diagonally by a plane, and one half being filled with a fluid is placed with the bisecting plane parallel to the horizon, and the vertex downwards. Find the time of emptying through a small orifice at the vertex.

Let $a = AB = AD$, and $x = Bg$;

$\therefore$ $ac = 2x$, and the descending surface $= 2ax$.

$$\therefore\ t = \int -\frac{2ax\,dx}{O\sqrt{2gx}} = -\frac{4a}{3O\sqrt{2g}} . x^{\frac{3}{2}} + C;$$

and $t = 0$, when $x = BG = \dfrac{a}{\sqrt{2}}$,

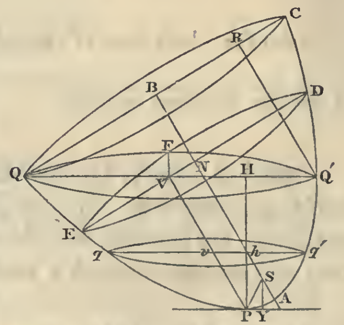


$$\therefore\ 0 = -\frac{4a}{3O\sqrt{2g}} \times \left(\frac{a}{\sqrt{2}}\right)^{\frac{3}{2}} + C,$$

$$\text{whence } t = \frac{4a}{3O\sqrt{2g}} \cdot \left\{ \left(\frac{a}{\sqrt{2}} \right)^{\frac{3}{2}} - x^{\frac{3}{2}} \right\},$$

$$\text{and the whole time} = \frac{2^{\frac{3}{4}} a^{\frac{5}{2}}}{3O\sqrt{g}}.$$

293. A hollow paraboloid has its axis inclined to an horizontal plane, till the surface of the fluid contained in it passes through one extremity of its base. In this position a hole being made at the extremity of that diameter which passes through the centre of the fluid's surface, and the greatest perpendicular that can be drawn from the surface of the fluid to the base being given; to determine the time of emptying.



Let QFQ' be the surface of the fluid, which therefore is an ellipse, and $QV = VQ'$. Let DFE be a section of the paraboloid perpendicular to the axis, and FV its section with the ellipse; therefore QV, VF are the semi-axes major and minor of the ellipse. Let VP be the diameter through V the centre of the surface; P is the lowest point of the paraboloid; since a tangent at P is parallel to $Q'Q$, and therefore to the horizon. From Q' draw $Q'R$ perpendicular to QC; it is the greatest perpendicular that can be drawn from the surface of the fluid to the base, and is therefore given. Let $l =$ the latus rectum, and $\theta =$ the angle of inclination of the axis to the horizon.

Then $RB^2 = l \times (AB - RQ')$; and $\therefore$ RB is known.

And BQ is given; $\therefore$ also RQ, and $\frac{RQ}{RQ'}, = \tan\theta$, may be found, whence the angle θ may be determined.

And (Euclid, I. 47.) QQ', and $\therefore$ QV also determined.

Again $VF^2 = DV \times VE = NE^2 - NV^2 = l \times PV$; whence

$$VF^2 : QV^2 (= 4SP \times PV) :: l : 4SP :: SY^2 : SP^2 :: \sin^2\theta : 1;$$

$$\therefore \; VF^2 = QV^2 . \sin^2\theta;$$

$$\text{and } \; PV = \frac{VF^2}{l} \text{ is known};$$

and $\therefore$ also PH which is $= PV \times \sin\theta$.

Let $PH = h$, $QV = a$, $VF = a . \sin\theta$; and let qfq' be any section parallel to the surface, and therefore similar to QFQ';

$$\therefore qfq' : QFQ' = \pi a^2 . \sin\theta :: qv^2 : QV^2 :: Ph : PH :: x : h;$$

$$\therefore \; qfq' = \pi a^2 . \sin\theta . \frac{x}{h};$$

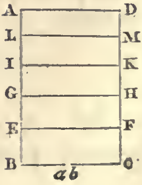
$$\text{and } t = \int -\frac{\pi a^2 . \sin\theta . x\,dx}{hO\sqrt{2gx}} = -\frac{\pi a^2 . \sin\theta}{hO\sqrt{2g}} . \tfrac{2}{3} x^{\frac{3}{2}} + C,$$

and when $t = 0$, $x = h$,

$$\therefore \; t = \frac{2\pi a^2 \sin\theta}{3hO\sqrt{2g}} . \{h^{\frac{3}{2}} - x^{\frac{3}{2}}\}.$$

$$\text{And the whole time} = \frac{2\pi a^2 . \sin\theta . h^{\frac{1}{2}}}{3O\sqrt{2g}}.$$

294. If a prismatic vessel contain fluids of different specific gravities which do not mix; to determine the time of emptying it through a small orifice in the base.



Let $ABCD$ be the vessel, containing any number of fluids whose upper surfaces are EF, GH, IK, LM, AD, &c. and whose respective specific gravities are s, s', s'', s''', &c. Now the pressure which produces the discharge of the fluid through the orifice, is the same as if there were substituted in the place of the upper fluids, another whose specific gravity is s, and having an altitude $\frac{s' \times GE}{s} + \frac{s'' \times IG}{s} + \frac{s''' \times LI}{s} +$ &c. (69). The problem then is reduced to that of determining the time of emptying a quantity of fluid $BEFC$, whose altitude at first $= BE + \frac{s' \times EG}{s} + \frac{s'' \times GI}{s} + \frac{s''' \times IL}{s} +$ &c. Call this altitude H; then (263) $T = \frac{2S}{O\sqrt{2g}} \times (\sqrt{H} - \sqrt{x})$, where x is a variable distance. And in order to determine the time in which the lower fluid is discharged; or the time in which the imaginary upper surface has descended through a space EB, made $x = \frac{s' . EG}{s} + \frac{s'' . GI}{s} + \frac{s''' \times IL}{s} +$ &c.

The lower fluid being now discharged, in place of the upper fluids HI, IM, &c. substitute one whose specific gravity is s', and whose altitude is $\frac{s'' . GI}{s'} + \frac{s''' . IL}{s'} +$ &c.;

$$\text{and making } H' = EG + \frac{s'' \times GI}{s'} + \frac{s''' \times IL}{s'} + \text{\&c.}$$

$$\text{and } x' = \frac{s'' \times GI}{s'} + \frac{s''' \times IL}{s'} + \text{\&c.}$$

$\therefore$ T', the time of emptying $FG = \frac{2S}{O\sqrt{2g}} . (\sqrt{H'} - \sqrt{x'})$.

And proceeding in a similar manner, whatever be the number of fluids,

$$\text{the whole time} = T + T' + T'' + T''' + \text{\&c.}$$

295. A prismatic vessel of given dimensions, with its sides vertical, is filled with fluid; there are two given and equal

small orifices, one at the bottom, the other bisecting the altitude. To find the time of emptying the upper half, supposing both orifices to be opened at the same instant.

Let $2a$ = the altitude of the vessel,

x = the altitude of the surface from the upper orifice at the end of t'',

r = the radius of the base.

Then $O\sqrt{2gx}$, and $O\sqrt{2g\,.\,(a+x)}$ are the quantities discharged in a second through the upper and lower orifices respectively;

$$\text{and} \quad O\sqrt{2g}\,.\,(\sqrt{x}+\sqrt{a+x})\,.\,dt = -\pi r^2 dx;$$

$$\therefore\ dt = -\frac{\pi r^2}{O\sqrt{2g}} \times \frac{dx}{\sqrt{x}+\sqrt{a+x}}$$

$$= -\frac{\pi r^2}{O\sqrt{2g}} \times \frac{\sqrt{a+x}-\sqrt{x}}{a}\,.\,dx;$$

$$\therefore\ t = -\frac{\pi r^2}{Oa\sqrt{2g}} \times \{\tfrac{2}{3}\,.\,(a+x)^{\frac{3}{2}} - \tfrac{2}{3}\,.\,x^{\frac{3}{2}}\} + C;$$

and $t=0$, when $x=a$;

$$\therefore\ 0 = -\frac{2\pi r^2}{3Oa\sqrt{2g}} \times \{(2a)^{\frac{3}{2}} - a^{\frac{3}{2}}\} + C;$$

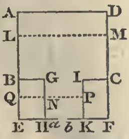
$$\therefore\ t = \frac{2\pi r^2}{3Oa\sqrt{2g}} \times \{(2\sqrt{2}-1)\,.\,a^{\frac{3}{2}} - (a+x)^{\frac{3}{2}} + x^{\frac{3}{2}}\};$$

and the time required when $x=0$, is

$$= \frac{2\pi r^2}{3Oa\sqrt{2g}} \times (2\sqrt{2}-2)\,.\,a^{\frac{3}{2}}$$

$$= \frac{4\pi r^2 a^{\frac{1}{2}}}{3O\sqrt{2g}}\,.\,\{\sqrt{2}-1\}.$$

296. If $ABCD$, $GHKI$ be two cylindrical or prismatic vessels filled with fluid; to determine the time in which the surfaces would descend through any given height, the fluid issuing through a very small orifice in the base of the lower.



Let A and B = the horizontal sections of the vessels.

The velocity at the orifice at each instant being that which is due to the depth below the surface, the fluid will issue, whilst the surface has not descended below BC, as it would through the same orifice in a vessel $AEFD$.

Hence the time down $AL = \dfrac{2A}{O\sqrt{2g}} \cdot (\sqrt{AE} - \sqrt{LE})$;

and the time down $AB = \dfrac{2A}{O\sqrt{2g}} \cdot (\sqrt{AE} - \sqrt{BE})$.

But when the surface has arrived at NP in the lower vessel,

the time down $GN = \dfrac{2B}{O\sqrt{2g}} \times (\sqrt{GH} - \sqrt{NH})$;

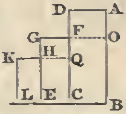
whence the time down AQ

$$= \frac{2}{O\sqrt{2g}} \cdot \{A . (\sqrt{AE} - \sqrt{BE}) + B . (\sqrt{GH} - \sqrt{NH})\}.$$

297. Cor. It is found by experiment that whilst the fluid is at a certain altitude above BC, the velocity at ab is that due to the depth, even if ab increases till it becomes $= HK$, provided HK is small compared with AD, and GH small. The time therefore in which AD descends to BC may be found as before. But when the surface arrives at GI (and $HK = ab$), $GHKI$ will descend as a heavy body, and with an initial velocity which is to the velocity at HK :: B : A

nearly. The time therefore of falling through *GH* will be found by the common formulæ.

298. If the vessels *ABCD, FCEG, HELK,* of any figure, be supposed to communicate by small apertures *C, E*; and the fluid to issue at *L*; to determine the heights due to the velocities at *C, E, L*; and the quantity issuing, when it becomes regular, that is, when the first receives as much as flows through *L*, and *AB, CF, EH* remain the same.



The orifices at *C, E, L* being small with respect to the dimensions of the vessels, the small column which passes through them will not disturb the continuity, at least sensibly. Produce *GF, KH* to *O* and *Q*. Then the columns *CFOB, CFGE* communicating through *C* will be in equilibrio; as also *CQHE, EHKL*; therefore the velocity at *C* is that due to *DF*, the velocity at *E* that due to *GH*, and the velocity at *L* that due to *KL*.

Let $AB = h$, $DF = x$, $GH = y$, $KL = z$; and *C, E, L* the areas of the apertures at *C, E, L* respectively. And let *Q* represent the quantity which flows through any of the orifices.

$$\therefore\ Q = Ct\sqrt{2gx} = Et\sqrt{2gy} = Lt\sqrt{2gz},$$

$$\text{and}\quad h = x + y + z;$$

$$\text{whence}\quad x = h \cdot \frac{E^2 L^2}{C^2 E^2 + C^2 L^2 + E^2 L^2},$$

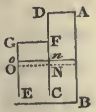
$$y = h \cdot \frac{C^2 L^2}{C^2 E^2 + C^2 L^2 + E^2 L^2},$$

$$z = h \cdot \frac{C^2 E^2}{C^2 E^2 + C^2 L^2 + E^2 L^2},$$

$$\text{and}\quad Q = Lt\sqrt{2gh} \times \frac{CE}{\sqrt{C^2 E^2 + C^2 L^2 + E^2 L^2}}.$$

299. If the vessel *ABCD* be kept constantly full to the height *CD*, and communicate with the prismatic vessel *FCEG*

by a small orifice at C; the fluid issuing through a small orifice at E: to determine the position of the surface of the fluid in $FCEG$ at the end of any given time.



Let NO be the position at the end of t'', and no the next successive position.

Let $CD = h$, $DN = x$, $A =$ the area of a section of CG.

During the time dt, a quantity flows through $C = Cdt\sqrt{2gx}$,

$$\text{and through } E = Edt\sqrt{2g.(h-x)};$$

$$\text{and the difference of these } = NOon,$$

$$\text{or } Cdt\sqrt{2gx} - Edt\sqrt{2g.(h=x)} = -Adx;$$

$$\text{hence } dt = \frac{A}{\sqrt{2g}} \times \frac{dx}{E\sqrt{h-x} - C\sqrt{x}}.$$

$$\text{Let } x = \frac{y^2}{h}, \text{ and } E\sqrt{h^2-y^2} - Cy = Ez;$$

$$\text{then } dt = Mdz + \frac{Nzdz}{\sqrt{P^2-z^2}} + \frac{Qdz}{z\sqrt{P^2-z^2}},$$

M, N, P, Q being constant quantities which are easily determined; whence t may be found in terms of x.

300. If $FCEG$ were not prismatic, A would not be constant, but some function of x and constant quantities depending upon the figure of the vessel; in which case the calculation would be more complicated.

301. At the commencement of the motion, the fluid in CG should have a certain altitude, in order that the fluid passing from $ABCD$ may not cause irregularity in the motion in CO. This condition will be satisfied by assuming, when $t = 0$, $DN =$ a given quantity a little less than h.

302. The computation becomes easy if we suppose that there is no orifice at E, but that the fluid which passes through C remains in the vessel CG.

In this case $E=0$, and $dt = \frac{A}{C\sqrt{2g}} \times \frac{-dx}{\sqrt{x}}$,

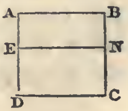
$$\therefore t = \frac{2A}{C\sqrt{2g}} \times -x^{\frac{1}{2}} + C'.$$

Let $x = h'$ when $t=0$;

then $t = \frac{2A}{C\sqrt{2g}} \times (\sqrt{h'} - \sqrt{x})$.

303. The time in which the fluid would be at the same level in the two vessels, will be $= \frac{2A}{C\sqrt{2g}} \times \sqrt{h'}$. This value is exactly the same as that of the time in which the vessel CG filled to an altitude h' would empty itself through an orifice $= C$ (263); that is, it is double the time in which, if kept constantly full to the altitude h', it would discharge a quantity Ah' through an orifice $= C$.

304. If a vessel $ABCD$ filled with fluid, be supplied at a given rate with fluid, a part of which issues through an orifice at D; to determine the position of its surface at any time; supposing the influx slower than the efflux when the vessel is full.



Since the quantity received is proportional to the time; it is evident that during the time dt it may be represented by $Odt\sqrt{2gb}$; O being a given orifice, and b the constant altitude due to the velocity at O.

Let NE be the position of the surface, and $CN = x$,

then $Ddt\sqrt{2gx} - Odt\sqrt{2gb} = -Adx$,

and $dt = \frac{-Adx}{\sqrt{2g}\cdot(D\sqrt{x} - O\sqrt{b})}$;

$$\therefore\ t = \frac{2A}{D\sqrt{2gb}} \cdot \left\{ \sqrt{bh} - \sqrt{bx} + \frac{Ob}{D} \cdot \text{h. l.} \left(\frac{\sqrt{bh} - \frac{Ob}{D}}{\sqrt{bx} - \frac{Ob}{D}} \right) \right\}$$

$$= \frac{2A}{D\sqrt{2g}} \cdot \left(\sqrt{h} - \sqrt{x}\right) + \frac{2AO\sqrt{b}}{D^2\sqrt{2g}} \times \text{h.l.} \left(\frac{D\sqrt{h} - O\sqrt{b}}{D\sqrt{x} - O\sqrt{b}} \right).$$

305. Cor. 1. The quantity which flows through D during the time t, is

$$= \int D\,dt\sqrt{2gx} = DA \,.\int \frac{-\sqrt{x}\,.\,dx}{D\sqrt{x} - O\sqrt{b}},$$

which may be determined in a manner similar to the above.

306. Cor. 2. It is evident that the surface of the fluid will not descend below that point where the rates of influx and efflux are equal, that is, where $D\sqrt{x} = O\sqrt{b}$,

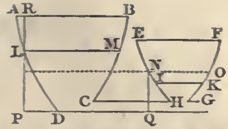
$$\text{or } x = \frac{O^2}{D^2} \cdot b.$$

307. Cor. 3. In this case the time =

$$\frac{2A}{D\sqrt{2g}} \cdot \left(\sqrt{h} - \frac{O}{D}\sqrt{b}\right) + \frac{2AO\sqrt{b}}{D^2\sqrt{2g}} \,.\ \text{h. l.}\ \frac{D\sqrt{h} - O\sqrt{b}}{O\sqrt{b} - O\sqrt{b}},$$

which is infinite: and consequently the surface of the fluid in the vessel can never arrive at the point where the rates of influx and efflux are equal. It appears also from Cor. 1, that the quantity discharged would be infinite.

308. A vessel $ABCD$ being filled to AB, is emptied through a small pipe which communicates with a second vessel $EFGH$ containing at first a portion of fluid whose upper surface is IK, and which flows through an orifice at G. If after a certain time the surfaces are at LM and NO; to determine the relation between their altitudes and the time.



Let $NQ = x$, and the surface $NO = X$, which will be given in terms of x, from the figure of the vessel. Let $LP = y$, and the surface $LM = Y$, which also will be known in terms of y from the figure of the vessel. Also let A and $B =$ the areas of the orifices.

The height due to the velocity at $H = y - x$;

$\therefore$ in the time dt the quantity issuing $= A\,dt\sqrt{2g\,.(y-x)}$,

which is $= -Ydy$;

$$\therefore\ dt = \frac{-Ydy}{A\sqrt{2g.(y-x)}}.$$

And the increase of the fluid in the vessel $EFGH$ is

$$A\,dt\sqrt{2g\,.(y-x)} - B\,dt\sqrt{2gx},$$

which is $= Xdx$;

$$\text{Hence } dt = \frac{Xdx}{\sqrt{2g}\,.(A\sqrt{y-x} - B\sqrt{x})};$$

$$\text{and } \therefore\ \frac{Xdx}{A\sqrt{y-x} - B\sqrt{x}} + \frac{Ydy}{A\sqrt{y-x}} = 0,$$

from which the relation of x and y must be determined; and thence the time.

But this cannot be done generally.

309. Cor. 1. If the vessel be kept full, or $y = a$, and be constant,

$$dt = \frac{Xdx}{\sqrt{2g}.(A\sqrt{a-x} - B\sqrt{x})};$$

from which the relation of t and x may be found.

310. Cor. 2. After some time the surface of the fluid in the second vessel will settle at a permanent level, as much fluid issuing as enters in an equal time. This permanent altitude will be found by making $dx = 0$;

$$\text{whence } A\sqrt{a-x}=B\sqrt{x},$$

$$\text{and } x=\frac{A^2a}{A^2-B^2}.$$

311. Cor. 3. When the vessels are prismatic, the equation (308) becomes homogeneous, and therefore the variable quantities are separable; X and Y being constant. Let them be M and N;

$$\text{then } \frac{Mdx}{A\sqrt{y-x}-B\sqrt{x}}+\frac{Ndy}{A\sqrt{y-x}}=0.$$

Suppose $y=zx$, and $z-1=u^2$,

$$\text{then } \frac{dx}{x}=\frac{2N.(Budu-Au^2du)}{NAu^3-NBu^2+(MA+NA).u-NB},$$

which being rational may be integrated.

312. Cor. 4. If whilst the fluid is passing from one vessel to another, none escapes; and the vessels are prismatical, $B=0$; then

$$Mdx+Ndy=0,$$

and making the whole vanish when $y=RP=a$, and $x=b$, a height also given;

$$\therefore\ Mx+Ny=Mb+Na;$$

whence

$$dt=\frac{-Ydy}{A\sqrt{2g.(y-x)}}=\frac{-N\sqrt{M}.dy}{A\sqrt{2g}.\sqrt{(M+N).y-Na-M}}$$

and t being $=0$, when $y=a$,

$$t=\frac{2N\sqrt{M}}{A.(M+N)\sqrt{2g}}.\{\sqrt{Ma-Mb}-\sqrt{(M+N).y-Na-Mb}\}.$$

313. Cor. 5. To determine the time when the fluid is on a level in the two vessels.

$$\text{Let } x=y=\frac{Na+Mb}{N+M};$$

$$\therefore t = \frac{2NM}{A.(M+N)} \cdot \frac{\sqrt{a-b}}{\sqrt{2g}}.$$

Also when $N = M$, $t = \frac{M}{A} \cdot \frac{\sqrt{a-b}}{\sqrt{2g}}$.

314. In the preceding cases the form of the orifice has not been considered; it being of little importance in horizontal apertures; but in each case it has been supposed to be small in comparison with the depth of the fluid.

If the orifice be lateral, and of such a magnitude that all its parts cannot be supposed equidistant from the surface of the fluid; it may be supposed to be divided by horizontal planes into an indefinite number of rectangles or trapeziums: and each of these being considered as a particular orifice, all of whose parts may be supposed equidistant from the surface of the fluid, the quantity which ought to be discharged in a given time may be determined (256). It will be only necessary then to find the sum of all these elementary quantities, in order to ascertain the whole quantity discharged from the entire orifice during the same time.

Prob. To determine the nature of an orifice to be made in the vertical side of a vessel which is always filled to the same altitude; so that the quantity discharged through any portion of it may be always as the height of that portion: and when the whole orifice is open, (c) cubic feet may be discharged every second; the top of the orifice being (a) feet, and the bottom (b) feet below the surface of the fluid; and the ordinates parallel to the horizon.

Let $y =$ the ordinate at the depth x,
and $m =$ that at the depth b.

Then the velocities at the depths x and b being $= \sqrt{2gx}$, and $\sqrt{2gb}$, the quantities discharged through the elementary trapeziums will be as $y\sqrt{2gx}$ and $m\sqrt{2gb}$. But as the quantity discharged varies as the height of the orifice, equal quantities will in the same time be discharged through each;

$$\therefore\ y\sqrt{x} = m\sqrt{b},$$

$$\text{and } y^2 x = m^2 b;$$

therefore the curve passing through the extremities of the ordinates will be an hyperbola of the second order.

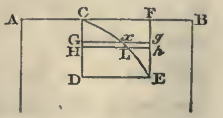
Also since the quantity discharged through each trapezium is the same, the whole quantity will be equal to that which would be discharged through an horizontal rectangular orifice at the depth b; the sides being $2m$ and $(b-a)$, that is, it will be $= 2m\,.(b-a).\sqrt{2gb}$;

but by the supposition, it is $= c$,

$$\therefore\ m = \frac{c}{2\sqrt{2gb}\,.\,(b-a)};$$

$$\text{and } y^2 x = m^2 b = \frac{c^2}{8g\,.\,(b-a)^2}.$$

Prob. If an orifice in the form of a parallelogram be made in the side of a vessel full of water; the quantity flowing through it in any given time will be two-thirds of the quantity that would flow through an equal orifice placed horizontally at the whole depth in the same time, the vessel being kept constantly full.



Let AB be the surface of the fluid in the vessel, $CDEF$ the given orifice; the velocity at D : that at any point H :: $\sqrt{CD} : \sqrt{CH}$. With abscissa CD therefore, and ordinate DE, describe a parabola CLE.

$$\text{then } V : v :: \sqrt{CD} : \sqrt{CH} :: DE : HL;$$

and if the abscissa be divided into an indefinite number of equal parts, as GH, or the area into an indefinite number of equal rectangles Gh, the quantity which flows through Gh : the quantity which flows through an equal orifice at the depth CD

$$:: GH \times HL : GH \times DE :: GxLH : GghH;$$

and as the same may be proved for all the other corresponding areas, the quantity which flows through the given orifice : the quantity through an equal orifice at the depth CD :: the sum of the areas $GxLH$: the sum of the areas $GghH$

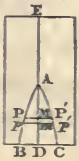
:: the area CDE : DF :: 2 : 3.

Cor. 1. If a and b be the altitude and breadth of the orifice,

the quantity discharged in any time t is $= \frac{2}{3} abt\sqrt{2ga}$.

Cor. 2. Hence the squares of the quantities discharged in the same time through the same orifice under different altitudes vary as the cubes of the altitudes.

315. To determine the relation between the time and the quantity of fluid issuing from a vessel through a vertical orifice; the vessel being kept constantly full.



Let ABC be a vertical orifice, formed by a plane curve, the vertex of which is A, and whose perpendicular depth from the surface of the fluid is AE.

Let $ED = h$, $EA = h'$, $AM = x$, $PP' = y$, PMP', pmp' being two horizontal ordinates indefinitely near to each other.

The area of the small trapezium $PP'p'p = ydx$,

and the velocity of the fluid being that due to the altitude $(h' + x)$, the quantity issuing through $PP'p'p$ in the time t, will be $= tydx\sqrt{2g.(h'+x)}$; and therefore the quantity (Q) issuing through the whole orifice in the time t, will be

$$= t\sqrt{2g}.(\int ydx\sqrt{h'+x} + C).$$

Now the nature of the curve which forms the orifice being given, the relation of x to y will be known; let therefore the value of y be substituted in the above equation, and the quantity discharged will be determined.

316. Cor. Since the velocity at the highest point of the orifice is the least, and increases from thence to the lowest point where it is the greatest: there must be some intermediate

depth where the velocity is such that if the whole column issued with the velocity due to that depth the quantity issuing in any time would be the same as through the given orifice.

This depth is called the *mean height*.

To determine this depth H;

The velocity at that depth $= \sqrt{2gH}$;

and the quantity issuing in the time t, $= t\sqrt{2gH}.(\int y\,dx + C')$.

Hence

$$t\sqrt{2gH}.(\int y\,dx + C') = t\sqrt{2g}(\int y\,dx\sqrt{h' + x} + C);$$

$$\text{and} \quad H = \frac{(\int y\,dx\sqrt{h' + x} + C)^2}{(\int y\,dx + C')^2}.$$

Ex. 1. Suppose the orifice to be a given rectangle.

Here y is constant, and

$$Q = ty\sqrt{2g}.(\int dx\sqrt{h' + x} + C) = ty\sqrt{2g}.\{\tfrac{2}{3}.(h' + x)^{\frac{3}{2}} + C\};$$

$$\text{and} \quad Q = 0 \quad \text{when } x = 0;$$

$$\therefore\ 0 = ty\sqrt{2g}.\{\tfrac{2}{3}h'^{\frac{3}{2}} + C\};$$

$$\therefore\ Q = \tfrac{2}{3}ty\sqrt{2g}.\{(h' + x)^{\frac{3}{2}} - h'^{\frac{3}{2}}\}.$$

And when $x = h - h'$, the whole quantity $= \frac{2}{3}ty\sqrt{2g}.\{h^{\frac{3}{2}} - h'^{\frac{3}{2}}\}$.

To find the *mean height*.

The area of the orifice being $= y.(h - h')$,

and the whole quantity effluent $= \frac{2}{3}ty\sqrt{2g}.\{h^{\frac{3}{2}} - h'^{\frac{3}{2}}\}$,

$$H = \frac{\{\frac{2}{3}(h^{\frac{3}{2}} - h'^{\frac{3}{2}})\}^2}{(h - h')^2} = \frac{4}{9}.\frac{(h^{\frac{3}{2}} - h'^{\frac{3}{2}})^2}{(h - h')^2}.$$

Cor. 1. If the upper side be on a level with the surface of a fluid,

$$H = \frac{4}{9}h = \frac{4}{9}a,$$

if a = the height of the rectangle.

$$\text{Also} \quad Q = \tfrac{2}{3}ty\sqrt{2g}.h^{\frac{3}{2}}.$$

Cor. 2. If the sum of the sides of the rectangle be equal to a given line, and the upper side be on a level with the surface of the fluid; to determine the value of the sides so that the quantity discharged through the whole orifice in a given time may be the greatest possible.

$$\text{In this case } \tfrac{2}{3}ty\sqrt{2g}.h^{\frac{3}{2}} = \text{max.};$$

$$\therefore\ yh^{\frac{3}{2}} = \text{max.},$$

$$\text{and } y + h = a;$$

$$\text{whence } h^{\frac{3}{2}}dy + \tfrac{3}{2}yh^{\frac{1}{2}}dh = 0,$$

$$\text{or } hdy + \tfrac{3}{2}ydh = 0;$$

$$\text{whence } y = \tfrac{2}{3}h;$$

$$\text{and } \therefore\ h + \tfrac{2}{3}h = a;$$

$$\text{or } h = \tfrac{3}{5}a;$$

$$\therefore\ y = \tfrac{2}{5}a.$$

Cor. 3. If the orifice be a trapezium with two horizontal sides respectively $=2p$ and $2q$, and altitude $=a$, the mean height might be found.

$$\text{For } \int ydx + C = a.(p+q),$$

$$\text{and } y = p + \frac{q-p}{a}.x,$$

which values being substituted, H may be found.

And if the upper side be on a level with the surface of the fluid,

$$H = \frac{16a}{225}.\frac{(2p+3q)^2}{(p+q)^2}.$$

Ex. 2. If the orifice be a triangle, whose vertex is upwards and base horizontal.

Let $a : b$ be the ratio of the altitude to the base,

$$\therefore\ y = \frac{bx}{a};$$

$$\text{and } Q = t\,.\,\frac{b}{a}\sqrt{2g}\,.\,\left(\int x\,dx\sqrt{h'+x} + C\right)$$

$$= t\,.\,\frac{b}{a}\sqrt{2g}\,.\,\left\{\tfrac{2}{5}\,.\,(h'+x)^{\frac{5}{2}} - \tfrac{2}{3}h'\,.\,(h'+x)^{\frac{3}{2}} + C\right\},$$

$$\text{and } Q = 0, \text{ when } x = 0;$$

$$\therefore\ 0 = -t\,.\,\frac{b}{a}\sqrt{2g}\,.\,\left\{\frac{4}{15}h'^{\frac{5}{2}} + C\right\},$$

$$\text{and } Q = t\,.\,\frac{b}{a}\sqrt{2g}\,.\,\left\{\frac{2}{5}\,.\,(h'+x)^{\frac{5}{2}} - \frac{2}{3}h'\,.\,(h'+x)^{\frac{3}{2}} + \frac{4}{15}h'^{\frac{5}{2}}\right\}$$

$$= \frac{2}{15}\,.\,t\,.\,\frac{b}{a}\sqrt{2g}\,.\,\left\{3h^{\frac{5}{2}} + 2h'^{\frac{5}{2}} - 5h'h^{\frac{3}{2}}\right\} \text{ when } x = h - h'.$$

Cor. 1. To find H,

$$\int y\,dx + C = \tfrac{1}{2}\,.\,\frac{b}{a}\,.\,(h-h')^2;$$

$$\therefore\ H = \frac{16\,.\,\{3h^{\frac{5}{2}} + 2h'^{\frac{5}{2}} - 5h'h^{\frac{3}{2}}\}^2}{(15)^2\,.\,(h-h')^4}.$$

Cor. 2. If the vertex coincide with the surface of the fluid,

$$H = \frac{16}{25}\,.\,h.$$

$$\text{And } Q = \frac{2}{15}t\,.\,\frac{b}{a}\sqrt{2g}\,.\,3h^{\frac{5}{2}}.$$

Ex. 3. If the vertex of the triangle be downwards and the base horizontal.

$$\text{Here } y = \frac{b}{a}\,.\,(h - h' - x),$$

$$\text{and } Q = t\,.\,\frac{b}{a}\sqrt{2g}\,.\,\left(\int (h-h'-x)\,.\,dx\sqrt{h'+x} + C\right)$$

$$= t\,.\,\frac{b}{a}\sqrt{2g}\left\{\int (h-h')\,.\,dx\sqrt{h'+x} - \int x\,dx\sqrt{h'+x} + C\right\}$$

$$= \frac{2}{15}t\,.\,\frac{b}{a}\sqrt{2g}\,.\,\left\{3h'^{\frac{5}{2}} + 2h^{\frac{5}{2}} - 5hh'^{\frac{3}{2}}\right\}, \text{ when } x = h - h'.$$

Cor. 1. To find H.

As before, $$H = \frac{16}{(15)^2} \cdot \frac{\{3h'^{\frac{5}{2}} + 2h^{\frac{5}{2}} - 5hh'^{\frac{3}{2}}\}^2}{(h-h')^4}.$$

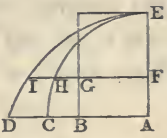
Cor. 2. If the base coincide with the surface of the fluid,

$$H = \frac{64}{225} \cdot h.$$

And $$Q = \frac{2}{15} t \cdot \frac{b}{a} \sqrt{2g} \cdot 2h^{\frac{5}{2}}.$$

Cor. 3. Hence the quantities discharged in the same time through two equal triangular orifices in the side of a vessel kept constantly full, the one having its base, and the other its vertex upwards, and their summits coinciding with the surface of the fluid, are in the ratio of 2 : 3.

Ex. 4. Three equal orifices of the same depth, are made in the side of a vessel; one a rectangular parallelogram, the second a semi-circle, and the third a parabola. To determine the ratio of the quantities of water which flow through them in the same time, the vessel being kept constantly full.



Let EA be the common altitude, and $= a$; $EF = x$; and let EB, ECA, EDA represent half the aperture of each.

Since the area of each $= \frac{1}{4}\pi a^2$;

$\therefore AB = \frac{1}{4}\pi a$, and $AD = \frac{3}{8}\pi a$.

And the quantity will $\propto \int y\,dx \sqrt{x}$;

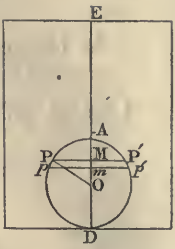
$$\therefore Q : Q' : Q'' = \int \tfrac{1}{4}\pi a \cdot x^{\frac{1}{2}} dx : \int FH \cdot x^{\frac{1}{2}} dx : \int FI \cdot x^{\frac{1}{2}} dx$$

$$= \frac{\pi a x^{\frac{3}{2}}}{6} : \int x\,dx \sqrt{2a - x} : \int \tfrac{3}{8}\pi \sqrt{a} \cdot x\,dx,$$

$$(\text{when } x = a) = \tfrac{1}{6}\pi a^{\frac{5}{2}} : \tfrac{8}{15} a^{\frac{5}{2}} \cdot (2\sqrt{2} - 7) : \tfrac{3}{16} \cdot \pi a^{\frac{5}{2}}$$

$$= 1 : \frac{16}{5\pi} \cdot (2\sqrt{2} - 7) : \frac{9}{8}.$$

Ex. 5. If the orifice be a circle.



Let the radius $AO=r$, and the height EO of the surface of the fluid above the centre of the orifice $=nr$; and let the angle $AOP=z$;

$\therefore\ PM=r.\sin z,\ \ OM=r.\cos z,$

and $EM=nr-r.\cos z;$

$\therefore\ Mm=d.AM=rdz\,.\sin z;$

$$\text{hence}\ \ PMmp=r^2.\sin^2 z.dz,$$

$$\text{and}\ Q=t\sqrt{2g}.\textstyle\int ydx.\sqrt{h'+x}$$

$$=2t\sqrt{2g}.r^2\textstyle\int\sin^2 z.dz\sqrt{nr-r.\cos z}$$

$$=2tr^2\sqrt{2gr}.\textstyle\int\sin^2 z.dz\sqrt{n-\cos z.}$$

Now $\int dz.\sin^2 z\sqrt{n-\cos z}=\int dz(1-\cos^2 z).\sqrt{n-\cos z}$
$=\int dz.(1-\cos^2.z).(n^{\frac{1}{2}}-\frac{1}{2}n^{-\frac{1}{2}}.\cos z-\frac{1}{8}.n^{-\frac{3}{2}}.\cos^2 z$
$-\frac{1}{16}n^{-\frac{5}{2}}.\cos^3 z-\text{\&c.}).$

And as the integral vanishes when $z=0$, and is complete when $z=360°$, the calculation will be lessened; as the quantities which involve cos. z, $\cos 2z$, &c. may be neglected, since the integrals contain $\sin z$, $\sin 2z$, &c. which vanish when $z=0$ or 360^0.

Now since $\cos^2 z=\frac{1}{2}.(1+\cos 2z)=\frac{1}{2}$, when $\cos z=0$,

$$\cos^3 z=\tfrac{3}{4}\cos z+\tfrac{1}{4}\cos 3z=0,$$

$$\cos^4 z=\tfrac{3}{8}+\tfrac{1}{2}\cos 2z+\tfrac{1}{8}\cos 4z=\tfrac{3}{8},$$

$$\cos^5 z=\tfrac{5}{8}\cos z+\tfrac{5}{16}\cos 3z+\tfrac{1}{16}\cos 5z=0,$$

$$\cos^6 z=\tfrac{5}{16}+\tfrac{15}{32}.\cos 2z+\tfrac{3}{16}\cos 4z+\tfrac{1}{32}.\cos 6z=\tfrac{5}{16},$$

$$\text{\&c.}=\text{\&c.}$$

$$\therefore\ Q=tr^2\sqrt{2gnr}.\textstyle\int dz.\left(1-\dfrac{1}{32n^2}-\dfrac{5}{1024n^4}-\text{\&c.}\right),$$

and making $z=2\pi$,

$$Q=tr^2\sqrt{2gnr}.2\pi.\left(1-\frac{1}{32n^2}-\frac{5}{1024n^4}-\text{\&c.}\right),$$

three terms of which series will be sufficient, since it converges rapidly.

Cor. 1. The mean height $= nr.\left(1 - \frac{1}{16}n^2 - \frac{9}{1024n^4} - \&c.\right).$

Cor. 2. When the surface of the water just touches the extremity A of the diameter AD, $n = 1$, and the preceding formula gives the quantity discharged in a given time.

It may also be found in finite terms as follows:

$$\text{Since } Q = 2tr^2\sqrt{2gr}.\int dz \sin^2 z\sqrt{1 - \cos z}.$$

$$\text{Let } 1 - \cos z = y; \quad \therefore \; 1 + \cos z = 2 - y;$$

$$\text{and } dz \sin^2 z\sqrt{1 - \cos z} = y\,dy\sqrt{2 - y};$$

$$\text{and } \int y\,dy\sqrt{2 - y} = -\tfrac{2}{3}y.(2 - y)^{\frac{3}{2}} + \tfrac{2}{3}\int(2 - y)^{\frac{3}{2}}dy$$

$$= -\tfrac{2}{3}y.(2 - y)^{\frac{3}{2}} - \tfrac{4}{15}.(2 - y)^{\frac{5}{2}} + C$$

$$= -\tfrac{2}{3}\sin^2 z.\sqrt{1 + \cos z} - \tfrac{4}{15}.(1 + \cos z)^{\frac{5}{2}} + C;$$

$$\therefore \; Q = 2tr^2\sqrt{2gr}.\left\{-\tfrac{2}{3}\sin^2 z\sqrt{1 + \cos z} - \tfrac{4}{15}.(1 + \cos z)^{\frac{5}{2}} + C\right\}:$$

but $Q = 0$ when $z = 0$;

$$\therefore \; 0 = 2tr^2\sqrt{2gr}.\left(-\frac{16\sqrt{2}}{15} + C\right);$$

whence

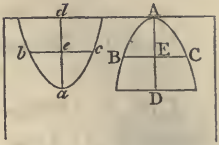
$$Q = 2tr^2\sqrt{2gr}.\left\{\frac{16\sqrt{2}}{15} - \frac{2}{3}\sin^2 z\sqrt{1 + \cos z} - \frac{4}{15}.(1 + \cos z)^{\frac{5}{2}}\right\};$$

and when $z = 180^0$,

$$Q = 2tr^2\sqrt{2gr}.\frac{16\sqrt{2}}{15} = \text{the quantity through } APD;$$

and for the whole orifice $Q = 2tr^2\sqrt{gr}.\frac{64}{15}$.

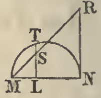
Ex. 6. Compare the quantities of fluid discharged by two equal parabolas in the side of a reservoir kept constantly full; one of them having its base, and the other its vertex downwards, and the summits of both coinciding with the surface of the fluid.



Let AD, ad be the axes of the parabolas. Draw the double ordinates CB, cb at the same depth; then the velocity at CB : that at cb :: $\sqrt{AE} : \sqrt{de}$.

Hence the quantities discharged in equal times through CB and cb are in the proportion of $CB : cb :: \sqrt{AE} : \sqrt{ae}$

$$:: AE : \sqrt{AE \times DE}.$$



Take $\therefore$ $MN = AD$, $ML = AE$; draw NR perpendicular and equal to MN; join MR, and draw the ordinate LST; MTN being a semicircle.

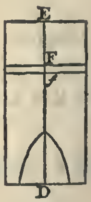
The quantity through CB : that through cb

$$:: ML : \sqrt{ML \times LN} :: LS : LT;$$

therefore the quantity through ACD : that through acd

$$:: MNR : MTN :: 4 : \pi.$$

317. If a vessel empty itself through a vertical orifice; to determine the time in which the surface will descend through a given height.



When the orifice is opened, let the surface of the fluid in the vessel be at E; and at the end of the time (t) at F. Let $EF = z$.

If during the time (t) the vessel were kept constantly full to F, the quantity discharged would be (315) $= t\sqrt{2g}\,.\,(\int y\,dx\sqrt{x + h' - z} + C)$. In this expression z is supposed constant, and therefore the integral may easily be found, and C determined according to the required conditions, viz. that when $x = 0$, $Q = 0$; and the complete integral, determined by making $x = h - h'$, will contain only z and known quantities.

Suppose now that a portion of fluid issues during the indefinitely small time dt, and the surface descends through $Ff = dz$. And therefore if $A =$ the area of the section of the vessel, the quantity issuing $= A\,dz$. But the surface may be supposed to descend with an uniform velocity through Ff, and the velocity at each point of the orifice to remain the same as when the surface was at F; hence the quantities issuing will be as the times; and

$$\therefore\ t\sqrt{2g}\,.\left(\int y\,dx\sqrt{x+h'-z}+C\right) : A\,dz :: t : dt;$$

$$\therefore\ dt = \frac{A\,dz}{\sqrt{2g}\,.\left(\int y\,dx\sqrt{x+h'-z}+C\right)};$$

$$\text{and } \therefore\ t = \frac{1}{\sqrt{2g}}\,.\int\frac{A\,dz}{\int y\,dx\sqrt{x+h'-z}+C}+C'.$$

The form of the vessel being known, A is also known in terms of z and constant quantities, and the integral in the denominator contains only z and invariable quantities; and therefore the value of t may be ascertained, and C' determined by taking $t = 0$ when $z = 0$.

Ex. 1. Let the vessel be a vertical prism, and the orifice a parallelogram.

In this case A and y are both constant;

$$\text{and} \int y\,dx\sqrt{x+h'-z} = \tfrac{2}{3}y\,.(x+h'-z)^{\frac{3}{2}}+C,$$

if z be supposed invariable.

Now this $=0$ when $x = 0$;

$$\therefore\ 0 = \tfrac{2}{3}y\,.(h'-z)^{\frac{3}{2}}+C;$$

$$\therefore \text{ the integral } = \tfrac{2}{3}y\,.\left\{(x+h'-z)^{\frac{3}{2}}-(h'-z)^{\frac{3}{2}}\right\}.$$

And when $x = h - h'$, it becomes $= \frac{2}{3}y\,.\left\{(h-z)^{\frac{3}{2}}-(h'-z)^{\frac{3}{2}}\right\}$;

$$\text{whence } t = \frac{3A}{2y\sqrt{2g}}\,.\int\frac{dz}{(h-z)^{\frac{3}{2}}-(h'-z)^{\frac{3}{2}}}.$$

Let $h - z = \psi$; $dz = -d\psi$;

and if $a =$ the height of the orifice, $h = h' + a$;

$$\therefore\ t=\frac{3A}{2y\sqrt{2g}}\times\int\frac{-d\psi}{\psi^{\frac{3}{2}}-(\psi-a)^{\frac{3}{2}}}$$

$$=\frac{3A}{2y\sqrt{2g}}\cdot\frac{-\psi^{\frac{3}{2}}d\psi}{3a\psi^2-3a^2\psi+a^3}-\frac{(\psi-a)^{\frac{3}{2}}.d\psi}{3a\psi^2-3a^2\psi+a^3},$$

multiplying numerator and denominator by $\psi^{\frac{3}{2}}+(\psi-a)^{\frac{3}{2}}$.

If $\psi=\phi^2$, and $\psi-a=\chi^2$, the expression will be rereduced to forms which are easily integrated; and when $\psi=h$, $t=0$; and the surface may be reckoned from $\psi=h$ to $\psi=a$.

Ex. 2. To find the time of emptying the prism by a vertical parallelogram reaching from the surface to the base.

In this case suppose $DF=x$, and the height of the fluid $=h$.

$$\text{The velocity at } F=\sqrt{2g\,.(h-x)};$$

and the quantity issuing through the orifice in the time t, supposing the vessel kept full,

$$=t\sqrt{2g}.y.\int-dx\sqrt{h-x}=\tfrac{2}{3}ty\sqrt{2g}.\{(h-x)^{\frac{3}{2}}+C\},$$

which determined between the values $x=0$ and $x=h$, gives

$$Q=\tfrac{2}{3}ty\sqrt{2g}.h^{\frac{3}{2}}.$$

$$\text{Hence } dt=\frac{-A\,dh}{\frac{2}{3}y\sqrt{2g}\,.\,h^{\frac{3}{2}}}=-\frac{3A}{2y\sqrt{2g}}\times\frac{dh}{h^{\frac{3}{2}}},$$

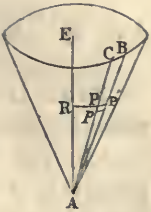
$$\text{and } t=\frac{3A}{2y\sqrt{2g}}\times\left\{\frac{2}{\sqrt{h}}+C\right\}=\frac{3A}{y\sqrt{2g}}\times\left(\frac{1}{\sqrt{h}}-\frac{1}{\sqrt{a}}\right),$$

if $a=$ the original height.

$$\text{And the whole time of emptying}=\frac{3A}{y\sqrt{2g}}\times\left(\frac{1}{0}-\frac{1}{\sqrt{a}}\right),$$

which is infinite.

Ex. 3. To find the time of emptying a cone by an orifice CAB in its side.



Let AE be the axis of the cone; $CB=b$, and $CA=l$; the angle $EAC=\alpha$, and $AR=x$, PR being perpendicular to AE.

$\therefore\ AP=x.\sec\alpha$, and $Pp=dx\ \sec\alpha$.

Let $h=$ the variable altitude of the fluid above A.

$$\text{Now } AC : CB :: AP : PP',$$

$$\text{or } l : b :: x\sec\alpha : PP';$$

$$\therefore\ PP'=\frac{b.\sec\alpha}{l}.x;$$

$$\therefore \text{ the area of } PpP'=-\frac{b.\sec\alpha}{l}.x\,dx,$$

and the quantity discharged

$$=-\frac{b.\sec\alpha}{l}.t\sqrt{2g}\int\sqrt{h-x}.x\,dx$$

$$=\frac{bt\sqrt{2g}}{l}.\sec\alpha.\frac{4}{15}h^{\frac{5}{2}},$$

taken between the values of $x=0$ and $x=h$.

$$\text{Hence } dt=\frac{-A\,dh}{\frac{4}{15}.\frac{bt\sqrt{2g}}{l}.\sec\alpha.h^{\frac{5}{2}}}$$

$$=\frac{-15l\pi.\tan^2\alpha.h^2dh}{4b\sqrt{2g}.\sec\alpha.h^{\frac{5}{2}}}$$

$$=-\frac{15l\pi.\tan^2\alpha}{4b\sqrt{2g}.\sec\alpha}.\frac{dh}{\sqrt{h}};$$

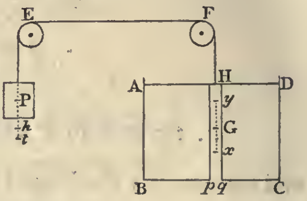
$$\therefore\ t=-\frac{15\pi l.\tan^2\alpha}{2b\sqrt{2g}.\sec\alpha}\times\sqrt{h}+C$$

$$=\frac{15\pi l.\tan^2\alpha}{2b\sqrt{2g}.\sec\alpha}.(\sqrt{a}-\sqrt{h}),$$

if $a=$ the original height.

And the whole time of emptying $= \frac{15\pi l . \tan^2 a}{2b\sqrt{2g} . \sec a} \times \sqrt{a}$.

318. If a vessel be filled with fluid to AD, and raised vertically by a weight P attached to a string, which will not stretch, and supposed without weight, passing over two pulleys E, F; to determine the pressure on a very small portion pq, or the height due to the velocity of the fluid issuing through pq.



Let G be the centre of gravity of the vessel and fluid contained in it, at any given time; M the whole mass. If P and M were acted on only by gravity and moved freely, they would describe the indefinitely small spaces (suppose) Pt, Gx in the same instant: but in consequence of their action on each other, let P describe Pk, and G describe Gy. The quantity of motion lost by the first being equal to that gained by the second in the same direction, according to D'Alembert's principle, let g represent the force of gravity, and f the simple accelerating force Pk or Gy,

$$P \times (g - f) = M \times (g + f);$$

$$\therefore f = g . \frac{P - M}{P + M};$$

$$\therefore xy = xG + Gy = g + \frac{g . (P - M)}{P + M} = \frac{2gP}{P + M},$$

which expresses the force that pushes up each particle of the mass M, in such a manner, that if an equal force were impressed in a contrary direction there would be an equilibrium. In this latter case there should act, against every point in the base, a pressure which is to the pressure that the same point would sustain if the fluid was acted upon only by gravity

$$:: \frac{2gP}{P + M} : g :: 2P : P + M.$$

Let $h = Hq$ the height of the fluid in the vessel; the height due to the velocity at pq would be $= h \times \frac{2P}{P+M}$; and this also represents the pressure of the fluid on each point of pq.

319. Cor. 1. Hence $dQ = Odt\sqrt{2g}.\sqrt{\frac{2hP}{P+M}}$.

320. Cor. 2. The vessel being supposed to empty itself through pq without receiving any fresh fluid, let $x =$ the variable altitude Hq, X the horizontal section of the vessel, which is a function of x, $M = C + \int X dx$;

$$\therefore\ -Xdx = Odt\sqrt{2g}.\sqrt{\frac{2Px}{P+C+\int Xdx}};$$

whence $dt = \frac{-Xdx.\sqrt{P+C+\int Xdx}}{O\sqrt{2g}.\sqrt{2Px}}$,

which will give the relation between t and x.

321. Cor. 3. If in the equation $f = g.\frac{P-M}{P+M}$, $P = M$, then $f = 0$,

$$\text{and } \frac{2P}{P+M} = 1.$$

Then the vessel is at rest, at least for an instant.

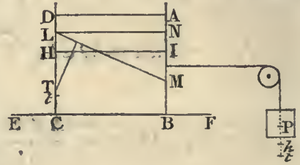
$$\text{Also } dQ = Odt\sqrt{2gh}.$$

322. Cor. 4. If $P = 0$, $\frac{2P}{P+M} = 0$;

therefore the pressure on $pq = 0$, or the fluid will not issue through the orifice. Which is also evident from this, that every particle in the mass M will descend by its own gravity, with the same velocity.

323. Cor. 5. If M be greater than P, and each of them be finite; M will descend and P ascend; therefore f becomes negative; but the velocity is such as would be acquired down a height $= h \cdot \frac{2P}{P+M}$.

324. If the vessel $ABCD$ containing a portion of fluid, be drawn along a horizontal plane by means of a weight P attached to a string, which is supposed to be without weight and not to stretch, passing over a pulley; to determine the position which the surface of the fluid will have when it has arrived at an uniform and permanent state; and the pressure on any point in the base.



Let M be the sum of the masses of the vessel and water; and Pt the space which the body would describe in an instant by its gravity, but in consequence of the action of M, let it only describe Pk, whilst M describes an horizontal equal space.

Let g represent the force of gravity, and f the accelerating force Pk.

By D'Alembert's principle $P \times (g - f) = Mf$,

$$\therefore f = \frac{g \cdot P}{P+M}.$$

Whence each particle is impelled in the direction EF by a force $\frac{gP}{P+M}$, and if an equal force were impressed in a contrary direction, there would be an equilibrium. In this case each particle of the fluid would be acted on by two forces, one vertical represented by g, the other horizontal represented by $\frac{gP}{P+M}$, and these would produce a compound force

$$g \cdot \frac{\sqrt{P^2 + (P+M)^2}}{P+M}.$$

In order then that there may be an equilibrium in the fluid, or that the fluid may have a fixed and permanent situation, the surface must be perpendicular to the direction of this force: and as it is always constant in quantity and direction, the surface of the fluid must be an inclined plane LM, such that drawing LN parallel to the horizon,

$$\frac{LN}{LM} = \frac{P+M}{\sqrt{P^2+(P+M)^2}}.$$

It is evident that LN and LM are given in magnitude.

The point L is determined from the figure of the vessel, and the quantity of fluid. For suppose (for instance) $ABCD$ to be rectangular; at rest, the horizontal surface is HI, and its altitude CH;

$$\therefore\ HC \times CB = \tfrac{1}{2}.(LC+BM).CB;$$

$$\text{whence } 2HC = LC + BM = 2LC - MN,$$

$$\text{and } LC = HC + \tfrac{1}{2}MN,$$

which is known, since $MN = \sqrt{LM^2 - LN^2}$ a known quantity.

If now from any point T in the side of the vessel TZ is drawn perpendicular to LM; the pressure against the indefinitely small area Tt, in this case, will be to the pressure against it, if at the depth TZ, and the fluid be at rest and acted on only by gravity

$$:: g.\frac{\sqrt{P^2+(P+M)^2}}{P+M} : g$$

$$:: \quad \sqrt{P^2+(P+M)^2} : P+M.$$

The pressure $\therefore$ will be $\dfrac{Tt \times TZ\sqrt{P^2+(P+M)^2}}{P+M}$.

325. Cor. Knowing the pressure on Tt, we know the velocity with which the fluid would issue at the instant of making a small orifice at Tt.

326. It may be observed that if the vessel loses no fluid, the surface will remain inclined, and will keep the same inclination as long as P moves with an uniformly accelerated motion. But if the vessel, having moved for some time with an uniformly accelerated motion, comes to rest or move uniformly, the surface will lose its inclination, and at length become horizontal. Which is also evident from the Proposition; for if $P = 0$, $f = 0$, and there being only a vertical force, the surface which is perpendicular to it, must become horizontal.

SECT. V.

327. DEF. BY the *Resistance* of fluids is meant that force by which the motions of bodies therein are impeded or retarded.

328. This resistance depends upon the inertia, the tenacity, and the friction of the fluid, supposing the particles to be in contact: but the two latter causes being very inconsiderable, we shall consider only the resistance arising from the inertia of the fluid.

329. The common hypothesis, upon which are founded the laws of this resistance, viz. that the fluid consists of an infinite number of parallel filaments in the direction of its motion, each of which communicates its impact on the body, without impeding the action of the others, is not true: the results deduced from it, in many cases differing from those which actual experiments give. It leads however to several curious conclusions, and may even be of service in examining the deductions to which we are led by experiments. Perhaps theory alone may not be adequate to meet the general question. Its deficiencies therefore must be supplied, and the formulæ, which it affords, modified by a series of actual experiments.

330. The force with which a stream impels a plane, is equal to the resistance suffered by the same plane when moving in the same direction, and with the same velocity through the fluid*.

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* Du Buat's experiments would lead to the inference that this is not precisely the case: the phænomena not being the same when the body is at rest as when it is in motion.

Since the motion communicated to a body B, at rest, by another body A impinging upon it with a certain velocity, is equal to the motion lost by B impinging with the same velocity upon A at rest: therefore the motion communicated to a plane by a stream impelling it with a certain velocity, is equal to the motion lost by the plane when moving in the same manner and with the same velocity through the fluid. But the former motion measures the force with which the stream impels the plane, and the latter measures the resistance suffered by the same plane, when moving in the same manner, and with the same velocity through the fluid.

331. Cor. Hence whatever is proved concerning the force of a stream impelling a plane, is true of the resistance suffered by that plane under the same circumstances.

332. If a stream impel a plane opposed to it perpendicularly; the force with which it impels the plane will be as the area (A) of the plane, the density (s) of the fluid, and the square of the velocity (v) of the stream, jointly.

For the force against the plane will be as the number of particles, and the force of each, jointly. And that number will $\propto A \times s \times v$; since of these three quantities any two being given, the number of particles impelling the plane will vary as the third. Also the force of each particle will $\propto v$; for the fluid being supposed to be homogeneous, the magnitude of each particle is the same. Hence therefore the number of particles × the force of each, that is, the whole force against the plane $\propto Asv^2$.

Here we suppose that after the particles strike the plane, the action of them ceases, whereas they must in reality diverge, and acting upon those which are behind affect their velocity: hence therefore a difference will arise between theory and experiment.

Ex. 1. If two streams of fluid whose specific gravities are as 5 : 1, moving with velocities which are as 2 : 3, fall perpendicularly on planes whose areas are as $5 : 4$; the forces will be as $5 \times 5 \times 4 : 1 \times 4 \times 9 :: 25 : 9$.

Ex. 2. If the specific gravities of water and air be as $900 : 1$; to determine the velocity of a stream of water, whose force may be equal to that of the wind blowing 3000 feet in a minute.

$$\text{Here } F : f :: (3000)^2 \times 1 : v^2 \times 900;$$
$$\therefore v^2 \times 900 = (3000)^2,$$
$$\text{and } v^2 = (100)^2; \therefore v = 100.$$

333. Cor. 1. The resistance will vary as the weight of a column of fluid whose base is the area of the plane, and altitude the space through which a body must descend, by the force of gravity, to acquire the velocity of the moving plane.

For the resistance $\propto Asv^2$; and the altitude of such a column would $\propto v^2$.

334. Cor. 2. Let $r =$ the radius of a cylinder, and $v =$ the velocity with which it moves in a fluid in the direction of its axis. Then the space through which a body must fall, by the force of gravity, to acquire the velocity (v) will be $= \frac{v^2}{2g}$; $\therefore$ the resistance to the cylinder $\propto \frac{Asv^2}{2g} \propto \frac{\pi r^2 sv^2}{2g}$; and if a be a constant quantity,

$$\text{the resistance} = \frac{\pi a r^2 s v^2}{2g}.$$

335. Cor. 3. If the weight of the cylinder $= w$; the retarding force, measured by the velocity destroyed in the cylinder in one second $= \frac{\pi a r^2 s v^2}{2w}$.

336. Cor. 4. If $l =$ the length of the cylinder, $s' =$ its specific gravity, $w = \pi r^2 l s'$; and therefore the retarding force $= \frac{asv^2}{2ls'}$.

337. If two fluids of the same kind impinge perpendicularly on two planes whose areas are A and a, with velocities V and v, and the planes themselves move in directions parallel to that of the fluid with velocities V' and v'; the impulsive forces, or the resistances will be as

$$A.(V \pm V')^2 : a.(v \pm v')^2,$$

according as the motions of the planes and fluid are in opposite or the same directions.

For the effect will be the same, whether the planes are supposed to be at rest and the fluid in motion, or the fluid at rest and the planes in motion. It is necessary therefore only to consider the relative velocities; which will be the sum or difference of the absolute velocities according as the motions are in opposite or the same directions.

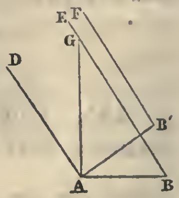
338. Cor. If one of the planes, as A, be at rest, $V' = 0$; and therefore the resistances will be as

$$AV^2 : a.(v \pm v')^2.$$

Hence therefore the impulsive force perpendicular to a plane at rest may be compared with that perpendicular to one in motion.

339. The force with which a stream impels a plane opposed to it obliquely $\propto \sin^2$ of inclination.

Let AB be a physical line opposed obliquely to a stream impelling it in the direction DA or EB; and AB' an equal physical line opposed perpendicularly to the same stream. Draw AG perpendicular to AB, and meeting BE in G.



Since the forces impelling AB' and AB are as the number of particles and force of each; and the numbers which impel AB' and AC are in the proportion of $AB : AC$, or rad : sin B; and the number which impel AC would, if suffered to proceed, impel AB in the same time; therefore the numbers (N and n) which impel AB' and AB are as rad : sin B. Take GB in the direction of the stream to represent the force of a particle against AB'. This may be resolved into two GA, AB; of which GA only will be effective on AB; hence if F be the force of a particle against AB', and f the effective force of a particle on AB,

$$F : f :: GB : GA :: \text{rad} : \sin B,$$

$$\text{and } N : n :: \quad \text{rad} : \sin B;$$

$\therefore$ the whole forces against AB' and AB are as rad^2 : sin^2 B.

And as the same may be proved of all other physical lines into which the two planes may be divided, the whole forces against the planes will be as rad^2 : sin^2 B; and the first and third terms being given, the force against the plane AB will vary as the sin^2 of inclination*.

340. Cor. 1. If ϕ be the force against the plane when perpendicular to the stream; the force impelling it obliquely $= \phi \cdot \sin^2 \theta$, if $\theta =$ the angle of inclination.

341. Cor. 2. If equal planes move in the same fluid and in directions equally inclined to themselves; the resistances are in the duplicate ratio of the velocities (332).

342. Cor. 3. If the same body, of any figure, moves in the same fluid with different velocities, and always keeps the same position in regard to the direction of its motion; the resistances will be in the duplicate ratio of the velocities.

343. Cor. 4. If different planes, differently inclined, move in different fluids, the resistances perpendicular to each plane will $\propto Asv^2 \cdot \sin^2 \theta$.

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* It appears from experiments, that the resistances which arise from oblique impulses do not vary as the sin^2 of the angles of inclination; but that when the angles are between 50° and 90° the common theory may be used as an approximation, observing that it always gives the resistances a little less than experiment, and as much less as the angles differ from 90°.

Ex. 1. A parallelogram and a triangle of the same base and altitude move in a fluid with velocities which are as 2 : 3. They are inclined to the directions of their motions at angles of 45^0 and 30^0 respectively. To compare the resistances on the planes in directions perpendicular to them.

The areas are as 2 : 1, the squares of the velocities as 4 : 9, and the $\sin^2$ of inclination as $\left(\frac{1}{\sqrt{2}}\right)^2 : \left(\frac{1}{2}\right)^2$;

$\therefore$ the resistances are as $\frac{8}{2} : \frac{9}{4} :: 16 : 9$.

Ex. 2. If a stream of fluid fall perpendicularly on a circular plane, and at an angle of 45° on an elliptical plane, whose minor axis is equal to the diameter of the circle, and impel these planes with equal forces; to determine the proportion of the major and minor axes of the ellipse.

Let x and r be the axes; the areas of the planes are as $r^2 : rx$, and the sines of inclination as $\sqrt{2} : 1$, the forces will be as $2r^2 : rx$; whence $x : r :: 2 : 1$.

344. The force with which a stream impels a plane in the direction of its motion $\propto \sin^3$ of inclination.

Proceeding as in the last proposition, (339), it may be shewn that the force, with which a particle impels AB in the direction perpendicular to the plane, will be represented by GA; which not being wholly effective in the direction of the stream, may be resolved into GC, CA; whereof CA being perpendicular to the direction of the stream, has no effect in communicating motion in that direction; therefore the force (F'), effective in the direction of the stream, is represented by GC; whence

$$F : F' :: GB : GC :: GB^2 : GA^2 :: \text{rad}^2 : \sin^2 B,$$

and (339) $N : n$ $:: \text{rad} : \sin B$,

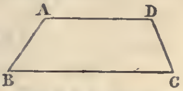
$\therefore$ the whole forces are as $\text{rad}^3 : \sin^3 B$.

Now the same being true for every other line into which each of the two planes may be divided, the forces against the planes are in that ratio; and the first and third terms being given, the force with which the stream impels the oblique plane in the direction of the stream $\propto \sin^3$ of inclination*.

345. Cor. 1. The force impelling the plane in the direction of the stream $= \phi \sin^3 \theta$.

346. Cor. 2. When different planes differently inclined, move in different fluids, their resistances in the directions of their motions $\propto A \times s \times v^2 \times \sin^3 \theta$.

Ex. A trapezium has two opposite sides equal, and the other two parallel. Compare the resistance upon it when it moves in the direction of the parallel sides, and when it moves in a direction perpendicular to them.



Let $AB = DC$; then the angle $ABC = DCB = \theta$, and let $R =$ the resistance on AB moving in the direction DA; and r and r' the resistances on AB and AD respectively, when moving perpendicularly;

$$\therefore R : r :: \sin^3 \theta : \text{rad}^3,$$

$$\text{but } r : r' :: AB : AD;$$

$$\therefore R : r' :: AB \times \sin^3 \theta : AD \times \text{rad}^3.$$

In the same manner if $R' =$ the resistance on AB or DC when moving perpendicularly to DA,

$$R' : r' :: AB \times \cos^3 \theta : AD \times \text{rad}^3;$$

$$\therefore 2R' + r' : r' :: 2AB \times \cos^3 \theta + AD \times \text{rad}^3 : AD \times \text{rad}^3;$$

$$\text{but } r' : R :: AD \times \text{rad}^3 : AB \times \sin^3 \theta;$$

$$\therefore 2R' + r' : R :: 2AB \times \cos^3 \theta + AD \times \text{rad}^3 : AB \times \sin^3 \theta.$$

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* By experiments on plane bodies moving both in air and water, the resistances have been found not to agree with the laws here deduced. Part of the difference *may* be occasioned by the tenacity of the fluid, and the friction of its particles against the body: but the greater part arises from the force after resolution, not taking effect, as is here supposed; the part which is considered not effective being *not all* lost.

347. The force with which a stream impels an oblique plane in a direction perpendicular to that of the stream $\propto \sin^2$ of inclination × cos of inclination.

It has been shewn (339.) (see Fig. 194) that the force with which a particle impels the oblique plane AB in a direction perpendicular to the plane is represented by GA; which not being wholly effective in a direction perpendicular to the stream may be resolved into GC, CA; of which GC, being in the direction of the stream, has no effect in communicating motion perpendicular to that direction; therefore the force (F'') effective in the proper direction is represented by CA; and

$$F : F'' :: GB : CA.$$

$$\text{Now } GB : GA :: \text{rad} : \sin B,$$

$$\text{and } GA : CA :: \text{rad} : \cos B;$$

$$\therefore F : F'' :: GB : CA :: \text{rad}^2 : \sin B \times \cos B,$$

$$\text{but } N : n :: \text{rad} : \sin B;$$

$$\therefore \text{the whole forces are as rad}^3 : \sin^2 B \times \cos B;$$

and, as before, the force which impels the plane in a direction perpendicular to the direction of the stream $\propto \sin^2 B \times \cos B$.

PROB. At what angle must the rudder of a vessel be inclined to the stream, that the effect produced may be a maximum?

$$\text{The effect} \propto \sin^2\theta \times \cos\theta = \text{max.}$$

$$\therefore \frac{d(\sin^2\theta)}{\sin^2\theta} + \frac{d\cos\theta}{\cos\theta} = 0,$$

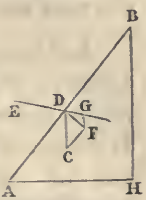
$$\text{or } \frac{2\sin\theta \times \cos\theta \times d\theta}{\sin^2\theta} - \frac{\sin\theta \times d\theta}{\cos\theta} = 0,$$

$$\text{and } 2\cos^2\theta = \sin^2\theta,$$

$$\text{or } 2 - 2\sin^2\theta = \sin^2\theta;$$

$$\therefore \sin\theta = \sqrt{\frac{2}{3}}.$$

PROB. To determine the angle CDA, which a plane AB must make with a stream impelling it in a given direction CD, so that the plane may be urged in another given direction DE with the greatest force possible.



Let CD represent the force of a particle of the fluid. Resolve it into CF parallel, and FD perpendicular to AB; of which FD alone is effective on the plane. Draw FG perpendicular to EDG; then FD may be resolved into FG, GD; of which GD only is effective in the given direction DE.

$$\text{Now } GD : FD :: (\cos GDF =) \sin ADE : \text{rad},$$

$$\text{and } FD : DC :: \sin ADC : \text{rad};$$

$$\therefore GD : CD :: \sin ADC \times \sin ADE : \text{rad}^2,$$

$$\text{and } GD \propto CD \times \sin ADC \times \sin ADE;$$

also the number of particles $\propto AH \propto \cos BAH \propto \sin ADC$; $\therefore$ the whole force impelling the plane in the given direction $\propto \sin^2 ADC \times \sin ADE$, which $=$ max.

$$\text{Also } CDA + ADE = CDE = \text{a given quantity};$$

$$\therefore dCDA = -dADE,$$

$$\text{and } \frac{d(\sin^2 CDA)}{\sin^2 CDA} + \frac{d \sin ADE}{\sin ADE} = 0,$$

$$\text{or } \frac{2 . \cos CDA \times dCDA}{\sin CDA} + \frac{\cos ADE \times dADE}{\sin ADE} = 0,$$

$$\text{whence } 2 . \tan ADE = \tan CDA;$$

$$\text{or } \tan ADE : \tan CDA :: 1 : 2;$$

$$\text{and } \tan ADE + \tan CDA : \tan ADE \sim \tan CDA :: 3 : 1,$$

$$\text{that is, } \sin CDE : \sin (CDA - ADE) :: 3 : 1;$$

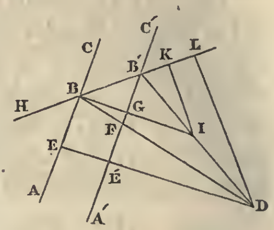
$$\therefore \sin (CDA - ADE) = \tfrac{1}{3} \sin CDE = \sin 2\beta \text{ (suppose)};$$

$$\therefore CDA - ADE = 2\beta;$$

$$\text{let } CDE = CDA + ADE = 2\alpha,$$

$$\left.\begin{array}{r} \therefore\ CDA = \alpha + \beta \\ \text{and } ADE = \alpha - \beta \end{array}\right\} \text{ which are both known.}$$

Prob. If a plane move with a velocity v, in a direction which makes an angle θ with the direction of the stream whose velocity is V; to determine the angle at which the plane must be inclined to the direction of its motion, so that the effect of the fluid may be the greatest possible.



Let the plane ABC move in the direction $B'B$ with the velocity v, which represent by $B'B$; and let DB represent the velocity and direction of the stream. Join DB'. The effect will be the same as if the plane were at rest (330) and acted upon by a fluid moving with a velocity DB'; therefore the force in direction perpendicular to the plane

$$\propto DB'^2 \times \sin^2 DB'A';$$

and its efficacy in the proper direction

$$\propto DB'^2 \times \sin^2 DB'A' \times \sin ABH, \text{ which } = \text{max.}$$

Now the angle DBB' being given, as also BB' and BD, the angle $BB'D$ and $B'D$ are also given, and therefore the efficacy is a max. when $\sin^2 DB'A' \times \sin ABH = \text{max.}$ that is, when $\tan DB'A' = 2 \,.\, \tan ABH$ (see last Prob.)

Draw BGI perpendicular to the plane. And since BG, GI are tangents of the angles ABH, $A'B'D$ to the same radius BG;

$$\therefore\ GI = 2GB.$$

Draw IK, DL perpendicular to HL, and let $GB = y$, $GB' = x$. From similar triangles $BB'G$, BIB', $BB' : BG :: BI : BK$,

$$\text{or } v : y :: 3y : BK = \frac{3y^2}{v};$$

$$\therefore\ B'K = BK - BB' = \frac{3y^2 - v^2}{v}.$$

$$\text{Also } BB' : B'G :: BI : IK,$$

$$\text{or } v : x :: 3y : IK = \frac{3xy}{v}.$$

And $DL = V . \sin\theta$, and $B'L = V . \cos\theta - v$.

But $DL : LB' :: IK : KB'$,

or $V . \sin\theta : V . \cos\theta - v :: \frac{3xy}{v} : \frac{3y^2 - v^2}{v}$,

$$\text{whence } 3y^2 - v^2 = 3xy . \frac{V . \cos\theta - v}{V . \sin\theta};$$

$$\text{but } v^2 = x^2 + y^2;$$

$$\therefore\ 2y^2 - x^2 = 3xy . \frac{V . \cos\theta - v}{V . \sin\theta} = 3xy . Q,$$

$$\text{and } \frac{x^2}{y^2} + 3Q . \frac{x}{y} = 2;$$

$$\text{whence } \frac{x}{y} = \sqrt{2 + \frac{9}{4} Q^2} - \frac{3}{2} Q$$

$$= \sqrt{2 + \frac{9}{4} . \left(\frac{V . \cos\theta - v}{V . \sin\theta}\right)^2} - \frac{3}{2} . \frac{V . \cos\theta - v}{V . \sin\theta},$$

which is the natural tangent of GBB', or the co-tangent of $GB'H$ the angle required.

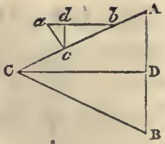
Cor. If the angle θ be a right angle, $\sin\theta = 1$, and $\cos\theta = 0$,

$$\text{whence } \frac{x}{y} = \sqrt{2 + \frac{9v^2}{4V^2}} + \frac{3}{2} . \frac{v}{V}.$$

And if the plane be at rest, or $v = 0$,

$$\frac{x}{y} = \sqrt{2}, \text{ which answers to } 54^\circ\ 44'.$$

Prob. If an isosceles wedge move through a fluid in the direction of its axis; compare the resistance on the sides with that on the base.



Since the same number of particles of fluid which strike against AC would, if suffered to proceed, strike AD in the same time, the ratio of the resistances will be the same as that of the force of a particle against each. Take ab to represent the force of a particle against AD; draw ac perpendicular to AC, and cd perpendicular to ab: then will ad be the effective force of a particle against AC in the direction of the axis; cd being destroyed by an equal and opposite force. Hence

$$\begin{aligned}\text{res. to } AD : \text{res. to } AC :: ab : ad &:: ab^2 : ac^2 \\ &:: AC^2 : AD^2 \\ &:: \text{rad}^2 : \sin^2 ACD;\end{aligned}$$

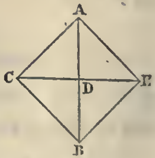
and the same being true for the other side and corresponding base,

the res. on the base : res. on the sides :: $\text{rad}^2 : \sin^2 \frac{1}{2} ACB$.

Cor. 1. If ACB be a right angle, the resistances are as $2 : 1$.

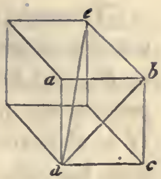
Cor. 2. The same may be proved in the case of a cone.

Prob. If a stream impel a cube in a direction perpendicular to one of its sides, and again in a direction perpendicular to one of its diagonal planes. Compare the forces.



The force perpendicular to AC : that perpendicular to AB :: $AC : AB :: 1 : \sqrt{2}$, and the force against AB : that against the sides :: $2 : 1$ (by the last Prob.); therefore the force perpendicular to AC : that against the sides (when moving in the direction of the diagonal) :: $2 : \sqrt{2} :: \sqrt{2} : 1$.

Prob. The resistance on a cube moving in a fluid in the direction of its diagonal is to the resistance on the same cube moving in the direction perpendicular to its side :: $1 : \sqrt{3}$.



Let $abcd$ be a side of the cube. Join bd, and make be perpendicular to the plane $abcd$, and $= ab$. Join de, which is therefore the diagonal of the cube, and the angle bde is its inclination to any one of the sides.

$$\text{Now } de^2 = db^2 + be^2 = da^2 + ab^2 + be^2 = 3be^2;$$

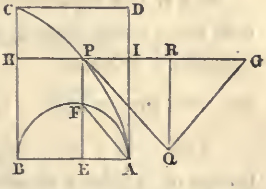
$$\therefore\ de = \sqrt{3} \, . \, be;$$

$$\text{and rad} : \sin bde :: de : eb :: \sqrt{3} : 1;$$

therefore the resistance on one side of the cube, moving in the direction of the diagonal, is to that (R) when it moves perpendicular to the side $:: 1 : 3\sqrt{3}$;

$\therefore$ the resistance on the three sides $: R :: 3 : 3\sqrt{3} :: 1 : \sqrt{3}$.

Prob. To compare the resistance on the curve of a cycloid moving in the direction of its axis, with that which would oppose the base.



Let ABC be the cycloid whose axis is AB. Complete the parallelogram $ABCD$. Draw GP parallel to AB, and let it represent the force of a particle perpendicular to the base. Draw PFE perpendicular to AB, GQ perpendicular to the tangent PQ, and QR perpendicular to PG. Then will GR represent the force of a particle on P. Join AF.

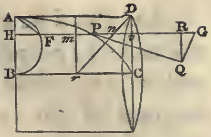
The force of a particle against H : that against P

$$:: GP : GR :: GP^2 : GQ^2$$

$$:: AF^2 : FE^2 :: BA : BE :: HI : HP.$$

Hence the whole resistances on the base and the curve are as the sum of all the HI's : the sum of all the HP's :: the parallelogram BD : the semi-cycloid ABC $:: 2 : \frac{3}{2} :: 4 : 3$.

Prob. A solid generated by the revolution of a common cycloid about its base, moves in a fluid in the direction of its base. To compare the resistance against this solid with that against its circumscribing cylinder.



Through any point P draw GPH parallel to the base CB; PQ a tangent at P; and taking GP, as before, to represent the force of a particle against H or the cylinder at v, GR will represent the effective force against P in the proper direction. Hence

$$\text{res. on } v : \text{res. on } P :: GP : GR :: GP^2 : GQ^2 :: AB^2 : AF^2$$
$$:: AB :: AH :: vm : vn,$$

if $vm = AB$, and $vn = AH$;

$\therefore$ res. on cylinder : res. on cycloidal solid :: sum of vm's : sum of vn's.

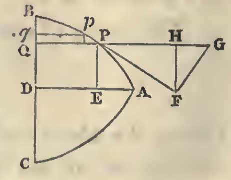
$$\text{Now } vm : vn :: AB : AH;$$
$$\therefore Cr : vn :: DC : Dv,$$

whence the locus of the point n is the straight line Dr.

But if the figure revolve round CB, the sum of all the vn's will be a cone, and the sum of the vm's its circumscribing cylinder; whence

res. on the cylinder : res. on the cycloidal solid :: 3 : 1.

348. If a plane figure, or a solid generated by the revolution of a plane figure round its axis, move in a fluid in the direction of its axis; to determine the ratio of the resistances on the curve or surface, and on the base.



Let BAC be a curve whose axis is AD, in the direction of which it moves. Take Pp a very small arc; and through P draw GPQ perpendicular to BD; and let GP represent the force of a particle against BD. Draw PF a tangent at P, and GF perpendicular to it, and FH perpendicular to GP; then will GH represent the effective force of a particle against P. Hence the force of a particle against Q : the effective force against P

$$:: GP : GH :: GP^2 : GF^2 :: dz^2 : dy^2.$$

Now if the figure be considered as a plane, the same number of particles which impel Pp, would, if suffered to proceed, impel Qq in same time, and $\propto Qq \propto dy$. Hence the whole force on Qq : the effective force on Pp

$$:: dz^2 \times dy : dy^3$$

$$:: dy : \frac{dy^3}{dz^2};$$

whence the whole force on the base : that on the curve

$$:: y : \int \frac{dy^3}{dz^2} :: y : \int \frac{dy}{1 + \frac{dx^2}{dy^2}}.$$

But if the figure be a solid generated by the revolution of a curve surface round AD, the number of particles impelling the surface, and the corresponding part of the base, will be proportional to the annulus generated by Qq or to

$2\pi y \times dy \propto y\,dy$, whence

the force on the base : that on the surface

$$:: \int y\,dy : \int \frac{y\,dy^3}{dz^2}.$$

$$:: \tfrac{1}{2}y^2 : \int \frac{y\,dy}{1 + \frac{dx^2}{dy^2}}.$$

Ex. 1. Let BAC be a semi-circle.

$$\text{Here } dz = \frac{r\,dy}{\sqrt{r^2 - y^2}};$$

and res. on base : res. on the curve

$$:: y : \int dy^3 \times \frac{r^2 - y^2}{r^2 dy^2}.$$

$$:: y : \int \frac{r^2 dy - y^2 dy}{r^2}$$

$$:: y : y - \frac{y^3}{3r^2},$$

which, when $y = r$, becomes as $r : r - \frac{1}{3}r :: 3 : 2$.

Ex. 2. Let BAC be a cone.

$$\text{Here } dy : dz :: BD : BA,$$

and res. on base : res. on surface $:: \frac{1}{2}y^2 : \int \frac{BD^2}{BA^2} \times y\,dy$

$$:: BA^2 : BD^2.$$

Cor. 1. If $R =$ the resistance on the base, the resistance on the surface $= R \,.\, \sin^2 BAD$.

Cor. 2. If the angle at the vertex be a right angle, the resistances are as $2 : 1$.

Prob. A right cone and a cylinder whose bases are the same, move in the same fluid in the direction of their axes; to determine the ratio of their velocities, when the resistances upon them are equal*.

Since when they move with the same velocity, the resistances are as $2 : 1$, and that *cæt. par.* the resistance $\propto V^2$;

$\therefore$ the res. on cylinder : res. on cone $:: 2V^2 : v^2$;

and $\therefore V : v :: 1 : \sqrt{2}$.

Ex. 3. Let BAC be a paraboloid.

Here $y^2 = 4ax$; $\therefore y\,dy = 2a\,dx$, and $\frac{dx^2}{dy^2} = \frac{y^2}{4a^2}$;

$\therefore$ res. on base : res. on surface $:: \frac{1}{2}y^2 : \int \frac{y\,dy}{1 + \frac{y^2}{4a^2}}$

$$:: \tfrac{1}{2}y^2 : 2a^2 \times \text{h.l.}(4a^2 + y^2).$$

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* If a cylinder moves in a fluid, so that the direction of its motion shall always coincide with the axis of the cylinder, the anterior plane surface only will communicate motion to the fluid; since the curved surface, being parallel to the direction in which the whole body moves, neither accelerates nor retards the particles of the fluid; these, as well as the cylinder, being supposed so smooth that no friction can have any sensible effect.

Ex. 4. Let *BAC* be a sphere.

The resistances will be $:: \frac{1}{2}y^2 : \int^1 \frac{(r^2 - y^2) . y\,dy}{r^2}$

$$:: \frac{1}{2}y^2 : \frac{1}{2}y^2 - \frac{y^4}{4r^2}$$

$$:: 2 : 1, \text{ when } y = r.$$

Cor. Hence the resistance to a sphere is half that to its circumscribing cylinder moving in the direction of its axis.

349. The force therefore retarding a sphere, measured by the velocity destroyed in one second, is $= \frac{\pi a r^2 s v^2}{4w}$, if $w =$ the weight of the sphere (335).

350. Also if $s' =$ the specific gravity of the sphere,

$$w = \frac{4\pi r^3 s'}{3};$$

$\therefore \frac{3asv^2}{16rs'} =$ the velocity generated or destroyed in the sphere in 1″ by a force equal to the resistance of the fluid, when the body moves with the uniform velocity v.

351. It is found by experiment*, that the resistance which a plane surface meets with, when directly and perpendicularly striking an indefinite fluid, is equal to the weight of a column of the fluid whose base is the area of the plane, and altitude the height due to its velocity. Suppose therefore $a = 1$, and $z =$ the space through which the sphere must

---

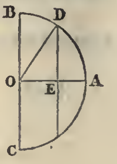
* If the fluid is not of indefinite extent, but merely a vein which strikes a plane surface at rest, the absolute measure of the resistance is different; being equal to a little less than the weight of a column of the fluid whose base is the area of the surface, and whose height is double the height which is due to the velocity of the issuing vein.

move with the velocity v, whilst a force equal to the resistance continued uniform, generates this velocity; then

$$v : z :: \frac{3sv^2}{16rs'} : v,$$

$$\text{or } s : s' :: \frac{16r}{3} : z.$$

Prob. Suppose a sphere to move in a resisting medium; it is required to cut off a segment, by a plane perpendicular to the direction of its motion, so that the resistance on the remaining frustum may be three-fourths of that on the end of a cylinder circumscribing the sphere.



Since

$$\text{res. on } DA : \text{res on } DE :: \tfrac{1}{2}y^2 - \frac{y^4}{4r^2} : \tfrac{1}{2}y^2$$
$$:: 2r^2y^2 - y^4 : 2r^2y^2,$$

$$\text{and res. on } DE : \text{res. on } OB :: y^2 : r^2,$$
$$\text{and res. on } OB : \text{res. on } AB :: 2 : 1;$$
$$\therefore \text{res. on } DA : \text{res. on } AB :: 2r^2y^2 - y^4 : r^4;$$

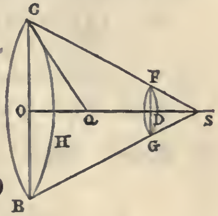
whence

$$\text{res. on } DA : \text{res. on } DB :: 2r^2y^2 - y^4 : r^4 - 2r^2y^2 + y^4;$$
$$\text{but since res. on } DE : \text{res. on } DA :: 2r^2y^2 : 2r^2y^2 - y^4;$$
$$\therefore \text{res. on } DE : \text{res. on } DB :: 2r^2y^2 : r^4 - 2r^2y^2 + y^4,$$
$$\text{and res. on } DE : \text{res. on } DE + \text{res. on } DB :: 2r^2y^2 : r^4 + y^4;$$
$$\text{but res. on } OB : \text{res. on } DE :: r^2 : y^2;$$
$$\therefore \text{res. on } OB : \text{res on } DE + \text{res. on } DB :: 2r^4 : r^4 + y^4;$$
$$\text{whence } 2r^4 : r^4 + y^4 :: 4 : 3,$$
$$\text{and } 2y^4 = r^4; \ \therefore \ y = r\sqrt[4]{\frac{1}{2}}.$$

Whence

$$AE = r - \sqrt{r^2 - y^2} = r - \sqrt{r^2 - \frac{r^2}{\sqrt{2}}} = r.\left(1 - \sqrt{1 - \frac{1}{\sqrt{2}}}\right).$$

Prob. Given the base and height of the frustum of a cone moving through a resisting medium; to find the whole altitude, so that the resistance on the frustum may be a min.



Let the resistance on the base $CHB = \pi . CO^2$, since the resistances on circles $\propto$ their areas; therefore that on $FDG = \pi . FD^2$; and the resistance on the surface of the frustum

$$= (\text{res. on } CHB - \text{res. on } FGD) . \sin^2 CSO$$

$$= \pi . (CO^2 - FD^2) . \frac{CO^2}{CS^2};$$

$\therefore$ the resistance on the frustum

$$= \pi . (CO^2 - FD^2) \frac{CO^2}{CS^2} + \pi . FD^2$$

$$= \pi . \frac{CO^4 + (CS^2 - CO^2) . FD^2}{CS^2}$$

$$= \pi . \frac{CO^4 + OS^2 . FD^2}{CS^2}$$

$$= \pi . \frac{CO^4 + CO^2 \times DS^2}{CS^2},$$

$$\text{whence } \frac{CO^2 + DS^2}{CS^2} = \text{min}.$$

Let $OD = a$, $OC = b$, $OS = x$;

$$\therefore \frac{b^2 + (x-a)^2}{b^2 + x^2} = \text{min}.$$

$$\text{and } \frac{(b^2 + x^2) \times 2 . (x-a) . dx - \{b^2 + (x-a)^2\} \times 2x dx}{(b^2 + x^2)^2} = 0;$$

$$\text{whence } x^2 - ax - b^2 = 0,$$

$$\text{and } x = \tfrac{1}{2}(a + \sqrt{a^2 + 4b^2}).$$

Hence the following construction:

Bisect OD in Q; join QC; produce OD, and make $QS = QC$; S is the vertex of the cone required.

For $CQ^2 = CO^2 + OQ^2 = b^2 + \frac{1}{4}a^2$;

$\therefore\ CQ = \sqrt{b^2 + \frac{1}{4}a^2}$;

and $OS = OQ + CQ = \frac{1}{2}a + \frac{1}{2}\sqrt{a^2 + 4b^2} = x$.

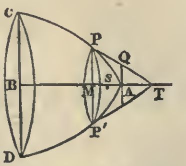
Cor. Since the angle $CQO = QCS + QSC$

$= 2QSC = CSB$;

$\therefore\ CSB$ is acute as long as CQO is.

When $\therefore\ Q$ comes to O, CQO, and $\therefore\ CSB$ are right angles; $\therefore$ the limit of QCS is 45^0, and then CFD will be its supplement, and $\therefore\ = 135^0$.

Prob. If CAB be a paraboloid, and from a point P a tangent be drawn forming with the tangent at the vertex a frustum of a cone; to determine the place of P, so that the resistance on $CPQA$ may be a min.



Let $AM = x$, $MP = y$; $\therefore\ TM = 2x$, and $y^2 = 4ax$.

Taking πy^2 or $4\pi ax$ to represent the resistance on the circle whose radius is MP; πax will represent that on the circle whose radius is AQ; $\therefore$ the resistance on the surface PQ

$$= (\text{res. on circle } MP - \text{res. on circle } AQ) \,.\, \sin^2 PTM$$

$$= 3\pi ax \,.\, \frac{PM^2}{PT^2}$$

$$= 3\pi ax \,.\, \frac{4ax}{4ax + 4x^2} = \frac{3\pi a^2 x}{a + x}.$$

Also the resistance on $AP = 2\pi \int y\,dy \,.\, \frac{dy^2}{dz^2}$

$$= 2\pi \int 2a\,dx \,.\, \frac{PM^2}{PT^2} = 2\pi \int 2a\,dx \,.\, \frac{a}{a + x};$$

$$\therefore \text{ the resistance on } CP = R - 2\pi\int\frac{2a^2\,dx}{a+x},$$

if R = the resistance on the whole paraboloid.

Hence the res. on $CPQA$

$$= R - 2\pi\int\frac{2a^2\,dx}{a+x} + \frac{3\pi a^2 x}{a+x} + \pi a x = \text{min.}$$

$$\therefore\ -\frac{4\pi a^2\,dx}{a+x} + \frac{3\pi a^2 . \{(a+x) . dx - x\,dx\}}{(a+x)^2} + \pi a\,dx = 0,$$

$$\text{or } 3\pi a^3 - 4\pi a^2 . (a+x) + \pi a . (a+x)^2 = 0,$$

whence $x = 2a$;

and if S be the focus, $AM = 2AS$.

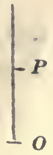
Also $y = 2\sqrt{ax} = 2a\sqrt{2}$;

$$\therefore \text{ tan. } MTP = \frac{MP}{MT} = \frac{y}{MT} = \frac{2a\sqrt{2}}{4a} = \frac{1}{\sqrt{2}}.$$

352. If a sphere of given magnitude be projected in a fluid whose specific gravity is to that of the sphere :: $s : s'$; having given the velocity of projection, to determine the velocity of the sphere at any given point; and the time of describing any given space.

Let O be the point from which the body is projected with the given velocity b.

Let $OP = x$, and the velocity at $P = v$.



$$\text{Then (350.) } \frac{3sv^2}{16rs'} \times dx = -v\,dv;$$

$$\therefore \frac{3s}{16rs'} . dx = -\frac{dv}{v};$$

$$\text{and } \frac{3s}{16rs'} . x = -\text{hyp. log. } v + C$$

$$= \text{hyp. log. } \frac{b}{v}.$$

Let e = the base of hyp. logs.; and $c = \frac{3s}{16rs'}$;

$$\therefore\ e^{cx} = \frac{b}{v},$$

$$\text{and}\ \ v = be^{-cx}.$$

$$\text{Also}\ \ dt = \frac{dx}{v} = \frac{e^{cx}dx}{b};$$

$$\therefore\ t = \frac{1}{bc}\cdot e^{cx} + C = \frac{1}{bc}\cdot\{e^{cx} - 1\},$$

since when $t = 0$, $x = 0$.

353. Cor. 1. If the spaces be taken in arithmetic progression, the velocities are in inverse geometric progression.

354. Cor. 2. If v be given, $x = \frac{1}{c}\cdot\left(\text{hyp. log. }\frac{b}{v}\right)$;

$$\text{and}\ \ t = \frac{1}{bc}\cdot\left(\frac{b}{v} - 1\right).$$

355. Cor. 3. If t be given, $v = \frac{b}{bct+1}$;

$$\text{and}\ \ x = \frac{1}{c}\cdot\{\text{hyp. log.}\,(bct+1)\}.$$

356. Cor. 4. To determine the space through which a body must move, so that the velocity at O may be to that at P in the ratio of $n : 1$.

In this case $b = nv$;

$$\text{but}\ \ v = be^{-cx} = nv\,.\,e^{-cx},$$

$$\text{whence}\ \ e^{cx} = n;\ \ \text{and}\ \therefore\ cx = \text{hyp. log}\ n,$$

$$\text{and}\ \ x = \frac{1}{c}\cdot(\text{hyp. log. } n) = \frac{16rs'}{3s}\times(\text{hyp. log. } n).$$

357. Cor. 5. Hence (355) $bct + 1 = n$,

$$\text{and} \quad t = \frac{n-1}{bc}.$$

358. Cor. 6. The velocity lost by a sphere projected in a fluid, during the time of its describing the space x, will be equal to that part of the initial velocity which is expressed by $\frac{e^{cx} - 1}{e^{cx}}$.

359. Cor. 7. When cx is very small, the velocity lost will approximate very nearly to cx.

For in this case $e^{cx} = 1 + cx$ very nearly.

360. If the sphere descend from rest; to determine its velocity at any point of its descent; and the time of descending through any given space.

The force (107) with which the sphere endeavours to descend in the fluid $= g \cdot \frac{s' - s}{s'}$;

and the retarding force of the fluid $= \frac{3sv^2}{16s'r}$;

therefore the whole force acting on the sphere is equal to the difference of these forces,

$$\text{or} \quad = g \cdot \frac{s' - s}{s'} - \frac{3sv^2}{16s'r};$$

$$\text{whence} \quad \left(g \cdot \frac{s' - s}{s'} - \frac{3sv^2}{16s'r}\right) \cdot dx = v\,dv,$$

$$\text{or} \quad \frac{3s}{8s'r} \cdot dx = \frac{2v\,dv}{16gr \cdot \frac{s' - s}{3s} - v^2};$$

$$\text{and } \therefore \frac{3s}{8s'r} \cdot x = -\text{hyp. log.} \left(16gr \cdot \frac{s' - s}{3s} - v^2\right) + C.$$

Now when $x = 0$, $v = 0$;

$$\therefore\ 0 = -\text{hyp. log.}\left(16gr.\frac{s'-s}{3s}\right)+C,$$

$$\text{whence}\ \frac{3s}{8s'r}.x = \text{hyp. log.}\ \frac{16gr.\dfrac{s'-s}{3s}}{16gr.\dfrac{s'-s}{3s}-v^2}$$

$$= \text{hyp. log.}\ \frac{16gr.(s'-s)}{16gr.(s'-s)-3sv^2};$$

$$\therefore\ e^{2cx} = \frac{16gr.(s'-s)}{16gr.(s'-s)-3sv^2};$$

$$\text{and}\quad e^{-2cx} = 1-\frac{3sv^2}{16gr.(s'-s)};$$

$$\therefore\ v^2 = \frac{16gr.(s'-s)}{3s}.(1-e^{-2cx}),$$

$$\text{and}\ v = \sqrt{\frac{16gr.(s'-s)}{3s}.(1-e^{-2cx})}.$$

$$\text{Let}\ m = \frac{16gr.(s'-s)}{3s},$$

$$\text{then}\ dt = \frac{dx}{v} = \frac{dx}{\sqrt{m}.\sqrt{1-e^{-2cx}}};$$

$$\therefore\ t = \frac{1}{\sqrt{m}}.\int\frac{dx}{\sqrt{1-e^{-2cx}}} = \frac{1}{2c\sqrt{m}}.\text{h. l.}\frac{1+\sqrt{1-e^{-2cx}}}{1-\sqrt{1-e^{-2cx}}};$$

since in this case $C=0$; t being $=0$, when $x=0$.

361. Cor. 1. If a heavy spherical body descend in a fluid, it is evident that it will every instant of its descent strike the fluid with the velocity acquired at that point; and that this impulsion, which from the nature of fluids is transmitted in all directions, will thus operate against the bottom of the vessel.

Now the resistance $= cv^2 = cm.(1 - e^{-2cx})$; and as e is greater than unity, e^{-2cx} decreases as x increases, and is $= 0$ when x is infinite; in this case therefore only, the impulsion on the bottom becomes $= cm = g.\frac{s' - s}{s'}$, which is the gravity of the body in the fluid (112).

This result is employed to illustrate the sinking of the barometer in rainy weather; for the distance from which the drops of rain fall is never very great, and therefore their action at the surface of the earth is always less than their absolute weight; whereas while they are held in solution they press with their actual weight; the mercury in the barometer therefore ought to fall during rain.

362. Cor. 2. If s' be indefinitely greater than s, then will $e^{-2cx} = 1 - 2cx$ very nearly;

whence $v = \sqrt{m.(1 - e^{-2cx})} = \sqrt{m.2cx} = \sqrt{2gx}$ = the velocity acquired whilst descending from rest by the acceleration of gravity through the space x.

363. Cor. 3. If the retarding and accelerating forces be equal,

$$\text{or } g.\frac{s' - s}{s'} = \frac{3sv^2}{16s'r},$$

the motion of the body in the fluid will be uniform, and the greatest that it can acquire.

$$\text{Then } v = \sqrt{16rg.\frac{s' - s}{3s}} = \sqrt{m}.$$

$$\text{But in general, } v = \sqrt{m.\frac{e^{2cx} - 1}{e^{2cx}}};$$

$$\therefore e^{2cx} - 1 = e^{2cx}, \text{ or } x \text{ is infinite.}$$

364. Cor. 4. This greatest velocity is equal to that acquired by a body in descending through the space $\frac{m}{2g}$ by

the force of gravity, or $\frac{s'}{s} \cdot \frac{8r}{3}$ by the relative weight of the body in the fluid*.

365. Cor. 5. Since $\frac{v^2}{m} = 1 - e^{-2cx}$;

$$\therefore \quad t = \frac{1}{2c\sqrt{m}} \text{ hyp. log. } \left(\frac{\sqrt{m}+v}{\sqrt{m}-v}\right)$$

$$= \frac{1}{2c\sqrt{m}} \cdot \text{hyp. log. } \left(1 + \frac{2v}{\sqrt{m}-v}\right).$$

If s' be indefinitely greater than s, $\sqrt{m}$ will be indefinitely greater than v when x is finite; $\therefore$ $\frac{2v}{\sqrt{m}-v}$ will be indefinitely small; and therefore

$$\frac{2v}{\sqrt{m}-v} = \text{hyp. log. } \left(1 + \frac{2v}{\sqrt{m}-v}\right),$$

$$\text{and } \therefore \; t \doteq \frac{1}{2c\sqrt{m}} \cdot \frac{2v}{\sqrt{m}-v} = \frac{v}{g},$$

the time of descent *in vacuo* to acquire the same velocity.

366. Cor. 6. If s' is less than s, the force with which the sphere endeavours to ascend in the fluid $= g \cdot \frac{s-s'}{s'}$. If therefore $v =$ the velocity which it will acquire in ascending through any altitude x, in a fluid of greater specific gravity,

$$v = \sqrt{m} \cdot \sqrt{1 - e^{-2cx}}; \text{ where } m = \frac{16gr \cdot (s-s')}{3s}.$$

$$\text{Also } t = \frac{1}{2c\sqrt{m}} \cdot \text{hyp. log. } \frac{1+\sqrt{1-e^{-2cx}}}{1-\sqrt{1-e^{-2cx}}}.$$

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* Wood's Mechanics (248). Whewell (178).

367. Cor. 7. If v be given, $x = \frac{1}{2c} . \text{h.l.} \left(\frac{m}{m - v^2}\right)$,

$$\text{and } t = \frac{1}{2c\sqrt{m}} . \text{hyp. log.} \left(\frac{\sqrt{m} + v}{\sqrt{m} - v}\right).$$

368. Cor. 8. If t be given, $v = \sqrt{m} . \frac{e^{2am^{\frac{1}{2}}t} - 1}{e^{2am^{\frac{1}{2}}t} + 1}$.

$$\text{Let } \frac{e^{2am^{\frac{1}{2}}t} - 1}{e^{2am^{\frac{1}{2}}t} + 1} = \phi,$$

$$x = \frac{1}{2c} . \text{hyp. log.} \left(\frac{m}{m - m\phi^2}\right) = \frac{1}{2c} \text{hyp. log.} \frac{1}{1 - \phi^2}.$$

369. If a sphere be projected *downwards* from a given point with a given velocity; to determine its velocity at any point of its descent; and the time of describing any space.

Let a = the velocity of projection.

From what has been shewn before, it appears that

$$\frac{3s}{8s''r} \times dx = \frac{2v\,dv}{16gr . \frac{s'' - s}{3s} - v^2};$$

$$\text{and } \therefore \frac{3s}{8s''r} . x = -\text{hyp. log.} \left(16gr . \frac{s'' - s}{3s} - v^2\right) + C$$

$$= -\text{hyp. log.} \frac{16gr . (s'' - s) - 3sv^2}{16gr . (s'' - s) - 3sa^2};$$

$$\therefore e^{-2ax} = \frac{16gr . (s'' - s) - 3sv^2}{16gr . (s'' - s) - 3sa^2},$$

$$\text{and } v^2 = \frac{16gr . (s'' - s) - 3sa^2}{3s} \times \left(\frac{16gr . (s'' - s)}{16gr . (s'' - s) - 3sa^2} - e^{-2ax}\right)$$

$$= (m - a^2) . \left[\frac{m}{m - a^2} - e^{-2ax}\right];$$

$$\text{whence } v=\sqrt{m-a^2}\,.\sqrt{\frac{m}{m-a^2}-e^{-2cx}}.$$

$$\text{Also since } \frac{3s}{8s'r}\,.\,dx=\frac{2v\,dv}{16gr.\dfrac{s'-s}{3s}-v^2};$$

$$\therefore\ dt=\frac{dx}{v}=\frac{8s'r}{3s}\,.\,\frac{2dv}{16gr\,.\dfrac{s'-s}{3s}-v^2}$$

$$=\frac{1}{2c}\,.\,\frac{2dv}{m-v^2}=\frac{1}{2c\sqrt{m}}\,.\,\frac{2\sqrt{m}\,.\,dv}{m-v^2};$$

$$\text{hence } t=\frac{1}{2c\sqrt{m}}\,.\,\text{hyp. log.}\ \frac{\sqrt{m}+v}{\sqrt{m}-v}+C.$$

But when $t=0$, $v=a$;

$$\therefore\ 0=\frac{1}{2c\sqrt{m}}\,.\,\text{hyp. log.}\left(\frac{\sqrt{m}+a}{\sqrt{m}-a}\right)+C;$$

$$\therefore\ t=\frac{1}{2c\sqrt{m}}\,.\,\text{hyp. log.}\left(\frac{\sqrt{m}+v}{\sqrt{m}-v}\times\frac{\sqrt{m}-a}{\sqrt{m}+a}\right).$$

$$\text{Now } v=\sqrt{m-(m-a^2)\,.\,e^{-2cx}};$$

$$\therefore\ t=\frac{1}{2c\sqrt{m}}\,.\,\text{h. l.}\left(\frac{\sqrt{m}+\sqrt{m-(m-a^2)\,.\,e^{-2cx}}}{\sqrt{m}-\sqrt{m-(m-a^2)\,.\,e^{-2cx}}}\times\frac{\sqrt{m}-a}{\sqrt{m}+a}\right);$$

hence when v or x is known, t may be found.

370. Cor. If the velocity with which the sphere is projected be equal to the greatest velocity it can acquire when descending from rest in the fluid, that is, if $a=\sqrt{m}$, the velocity will continue uniform.

$$\text{But } dt=\frac{dx}{v}=\frac{dx}{a};$$

$$\therefore\ \text{in this case } t=\frac{x}{a};$$

371. If the sphere be projected *upwards* with a given velocity; to determine its velocity at any point of its ascent; and the time of ascending through any given space.

In the preceding propositions it has been shewn that the force with which the sphere endeavours to descend in the fluid $= g \cdot \frac{s'-s}{s'}$; and the retarding force $= \frac{3sv^2}{16s'r}$.

In this case these forces act in the same direction, and

$$\therefore \left(g \cdot \frac{s'-s}{s'} + \frac{3sv^2}{16s'r}\right) . dx = -vdv,$$

$$\text{or } (cm + cv^2) . dx = -vdv;$$

$$\therefore 2cdx = -\frac{2vdv}{m+v^2};$$

$$\therefore 2cx = -\text{hyp. log. } (m+v^2) + C.$$

Now when $x = 0$, $v = a$;

$$\therefore 0 = -\text{hyp. log. } (m+a^2) + C;$$

$$\therefore 2cx = \text{hyp. log. } \left(\frac{m+a^2}{m+v^2}\right);$$

$$\text{and } v = \sqrt{m+a^2} . \sqrt{e^{-2cx} - \frac{m}{m+a^2}}.$$

$$\text{Again } dt = \frac{dx}{v} = -\frac{1}{2c} \cdot \frac{2dv}{m+v^2},$$

$$\text{and } \therefore t = \frac{-1}{c\sqrt{m}} \times \left(\text{circ. arc, tan } = \frac{v}{\sqrt{m}}\right) + C$$

$$= \frac{1}{c\sqrt{m}} \times \left(\text{difference of two circ. arcs whose tangents are } \frac{a}{\sqrt{m}} \text{ and } \frac{v}{\sqrt{m}}\right).$$

372. Cor. 1. If $v=0$, $x=\frac{1}{2c}\cdot\text{hyp. log.}\left(\frac{m+a^2}{m}\right)$

$$=\frac{1}{2c}\cdot\text{hyp. log.}\left(1+\frac{a^2}{m}\right),$$

which is the greatest height the body can ascend to with the velocity a.

373. Cor. 2. In this case if s' be indefinitely greater than s, then m will be indefinitely great, and $\therefore$ $\frac{a^2}{m}$ very small;

$$\text{hence hyp. log.}\left(1+\frac{a^2}{m}\right)=\frac{a^2}{m};$$

and therefore the height ascended in this case $=\frac{a^2}{2g}$,

the height through which the body will ascend *in vacuo* to lose the velocity a.

374. Cor. 3. To determine the velocity of the sphere when it returns again to the point of projection.

Let hyp. log. $\frac{m+a^2}{m}=z$;

the space through which the body descends from rest $=x$

$$=\frac{1}{2c}\cdot z,$$

$$\text{and } 2cx=z.$$

$$\text{But (360) the velocity}=\sqrt{m\cdot(1-e^{-2cx})}$$

$$=\sqrt{m\cdot(1-e^{-z})}.$$

375. Cor. 4. Since $z=\text{hyp. log.}\left(\frac{m+a^2}{m}\right)$,

$$e^{-z}=\frac{m}{m+a^2};$$

therefore the velocity of the sphere when it returns to the point of projection $= \sqrt{m - \frac{m^2}{m+a^2}} = a\sqrt{\frac{m}{m+a^2}}$.

If $\therefore$ s' be indefinitely greater than s, and $\therefore$ m indefinitely great, the velocity of the body when it returns to the point of projection $= a$, as it ought.

376. Cor. 5. The time of ascending to the greatest altitude, that is, till $v = 0$, is

$$= \frac{1}{c\sqrt{m}} \cdot \left(\text{circ. arc, whose tan} = \frac{a}{\sqrt{m}}\right).$$

Hence therefore (360) if $2cx = z$, the time of ascending and descending again

$$= \frac{1}{c\sqrt{m}} \cdot \left[\left(\text{arc, tan} = \frac{a}{\sqrt{m}}\right) + \tfrac{1}{2}\,\text{hyp. log.}\frac{1+\sqrt{1-e^{-z}}}{1-\sqrt{1-e^{-z}}}\right].$$

377. Cor. 6. Since $e^{-z} = \frac{m}{m+a^2}$;

$$\therefore \frac{1+\sqrt{1-e^{-z}}}{1-\sqrt{1-e^{-z}}} = \frac{\sqrt{m+a^2}+a}{\sqrt{m+a^2}-a} = 1 + \frac{2a}{\sqrt{m+a^2}-a},$$

hence the time of ascent and descent

$$= \frac{1}{e\sqrt{m}}\left[\left(\text{arc, tan} = \frac{a}{\sqrt{m}}\right) + \tfrac{1}{2}\,\text{hyp. log.}\left(1 + \frac{2a}{\sqrt{m+a^2}-a}\right)\right].$$

378. Cor. 7. When the ratio of $s' : s$ is *very great*, this time becomes $\frac{a}{\frac{1}{2}g}$, the same as *in vacuo*; as will appear by proceeding in the same manner as before.

379. A sphere moving in a fluid with a given velocity (k) meets with a resistance equal to the weight of a column of the fluid, having for its base the greatest section of the sphere and height (h). If the resistance $\propto (\text{vel.})^n$; to determine the velocity at any point of the body's motion, and the time of

describing any given space, when the sphere is projected with a given velocity.

Let b = the given velocity of projection, and R = the resistance which the sphere meets with, when moving with the velocity v;

$$\text{then} \quad k^n : v^n :: \pi r^2 hs : R;$$

$$\therefore \quad R = \pi r^2 hs \times \frac{v^n}{k^n};$$

$$\text{and the retarding force} = g \cdot \frac{R}{w} = g \cdot \pi r^2 hs \cdot \frac{v^n}{k^n} \cdot \frac{3}{4 \pi r^3 s'}$$

$$= \frac{3ghs}{4rs'k^n} \cdot v^n,$$

$$\text{whence} \quad \frac{3ghs}{4rs'k^n} \cdot -v^n dx = v\,dv,$$

$$\text{and} \quad \frac{3ghs}{4rs'k^n} \cdot dx = -\frac{dv}{v^{n-1}};$$

$$\therefore \quad \frac{3ghs}{4rs'k^n} \cdot x = \frac{1}{n-2} \cdot \frac{1}{v^{n-2}} + C$$

$$= \frac{1}{n-2} \cdot \left(\frac{1}{v^{n-2}} - \frac{1}{b^{n-2}}\right);$$

$$\text{whence} \quad v = \frac{b}{\left(1 + (n-2) \cdot b^{n-2} \cdot \frac{3ghsx}{4rs'k^n}\right)^{\frac{1}{n-2}}}.$$

$$\text{Let} \quad \phi = (n-2) \cdot b^{n-2} \cdot \frac{3ghs}{4rs'k^n},$$

$$\text{then } dt = \frac{dx}{v} = \frac{dx \cdot (1 + \phi x)^{\frac{1}{n-2}}}{b},$$

$$\therefore\ t = \frac{1}{b\phi} \times \frac{n-2}{n-1}.(1+\phi x)^{\frac{n-1}{n-2}} + C$$

$$= \frac{n-2}{n-1}.\frac{1}{b\phi}.\left\{(1+\phi x)^{\frac{n-1}{n-2}} - 1\right\}.$$

380. Cor. 1. If $n=1$, or $R \propto V$,

$$v = b - \frac{3ghsx}{4rs'k},$$

$$\text{and } x = \frac{4rs'k}{3ghs}.(b-v);$$

hence the diminution of velocity is proportional to the space described.

381. Cor. 2. If $n=1$, and $v=0$,

$$x = \frac{4rs'k}{3ghs}.b,$$

the greatest space the body can describe with the given velocity of projection, or the space described before its motion be destroyed.

382. Cor. 3. When $n=1$, $dt = \frac{dx}{v} = \frac{dx}{b - \frac{3ghs}{4rs'k}.x}$;

$$\therefore\ t = \frac{4rs'k}{3ghs} \times \text{hyp. log.} \left\{\frac{1}{b - \frac{3ghs}{4rs'k}.x}\right\} + C$$

$$= \frac{4rs'k}{3ghs} \times \text{hyp. log.} \left\{\frac{b}{b - \frac{3ghs}{4rs'k}.x}\right\}$$

$$= \frac{4rs'k}{3ghs} \times \text{hyp. log.} \left(\frac{4rs'kb}{4rs'kb - 3ghsx}\right).$$

383. Cor. 4. In this case if $x = \frac{4rs'kb}{3ghs}$, t will be in-

finite; therefore in any finite time the body will not lose all its velocity (381), nor describe the finite space $\frac{4rs'kb}{3ghs}$.

384. Cor. 5. Since when $n = 1$, $\frac{3ghs}{4rs'k} . dx = dv$;

$$\therefore\ dt = \frac{dx}{v} = \frac{4rs'k}{3ghs} \times -\frac{dv}{v},$$

$$\text{and}\ \ t = \frac{4rs'k}{3ghs} \times \text{hyp. log.} \frac{1}{v} + C$$

$$= \frac{4rs'k}{3ghs} \times \text{hyp. log.} \frac{b}{v};$$

$$\therefore\ v = b e^{-\frac{3ghs}{4rs'k} . t},$$

from which it appears, that the body will never cease to move forward, whatever be the density of the fluid.

385. Cor. 6. Hence also, if the times be taken equal, the velocities at the beginnings of those times are in geometrical progression.

386. Cor. 7. If the body be projected in a fluid, directly to or from a centre of force, and be attracted towards that centre by a constant force ϕ, or by a force which varies according to any given law of the distance; the velocity of the body, after any given space has been described, may be found; as also the time of describing it.

For when the body descends, the whole accelerating force $= \phi -$ the retarding force of the fluid, and when it ascends, the whole retarding force $= \phi +$ the retarding force of the fluid; whence

$$vdv = \pm(\phi \mp \phi') . dx,$$

in which equation the values of ϕ and ϕ' being substituted, the relation of v and x, and therefore of t and x may be found.

387. Cor. 8. If ϕ be constant, and the resistance as in (379), and $n = 2$,

$$\phi' = \frac{3ghs}{4rs'k^2} \times v^2 = \mu v^2;$$

$$\therefore\ vdv = \pm(\phi \mp \mu v^2) . dx,$$

$$\text{and}\quad dx = \frac{\pm v dv}{\phi \mp \mu v^2};$$

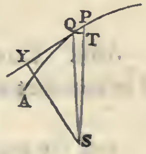
$$\text{whence}\quad x = -\frac{1}{2\mu} \times \text{hyp. log.}\ (\phi \mp \mu v^2) + C$$

$$= \frac{1}{2\mu} \times \text{hyp. log.}\ \left(\frac{\phi \mp \mu b^2}{\phi \mp \mu v^2}\right),$$

$$\text{and}\quad e^{2\mu x} = \frac{\phi \mp \mu b^2}{\phi \mp \mu v^2},$$

whence v may be found; and $\therefore$ t as before.

388. If a body revolve in a resisting medium round a centre of force, and the law be given according to which it varies; to determine the ratio of the resistance to the central force, so that the body may describe any given curve.



Let AQP be the curve in which the body moves round the centre of force S; Q the place of the body at any time. Let $SQ = y$, and $AQ = z$; and let $\gamma = \frac{1}{2}$ chord of curvature passing through S; and v = the velocity of the body at Q.

Then if ϕ = the centripetal force,

$\frac{\phi dy}{dz}$ = that part which acts in the direction of the curve or tangent;

if $\therefore$ F = the whole force in the direction of the tangent;

$$F = \phi' + \frac{\phi dy}{dz},$$

$$\text{or}\quad \phi' = F - \frac{\phi dy}{dz}.$$

$$\text{Now}\quad \phi = \frac{v^2}{\gamma},\ \text{or}\ v = \sqrt{\phi\gamma};$$

$$\therefore -dv = -\frac{\gamma d\phi + \phi d\gamma}{2\sqrt{\phi\gamma}};$$

$$\text{and } dt = \frac{dz}{v} = \frac{dz}{\sqrt{\phi\gamma}};$$

$$\therefore F \times \frac{dz}{\sqrt{\phi\gamma}} = -\frac{\gamma d\phi + \phi d\gamma}{2\sqrt{\phi\gamma}},$$

$$\text{or } F = -\frac{\gamma d\phi + \phi d\gamma}{2dz};$$

$$\therefore \phi' = -\frac{\gamma d\phi + \phi d\gamma}{2dz} - \frac{\phi dy}{dz}$$

$$= -\frac{\gamma d\phi + \phi d\gamma + 2\phi dy}{2dz};$$

$$\therefore \text{the centripetal force} : \phi' :: \phi : -\frac{\gamma d\phi + \phi d\gamma + 2\phi dy}{2dz}$$

$$:: 1 : -\frac{\gamma d\phi + \phi d\gamma + 2\phi dy}{2\phi dz}.$$

389. Cor. If the centre is at an infinite distance, ϕ is constant and acts in parallel lines; $\therefore d\phi = 0$;

$$\text{and the centripetal force} : \phi' :: 1 : -\frac{d\gamma + 2dy}{dz}.$$

$$\text{But } \gamma = -\frac{dz^2}{d^2y};$$

$\therefore$ substituting the value of $d\gamma$,

$$\text{the centripetal force} : \phi' :: 1 : -\frac{d^3y\,dz}{2d^2y^2}.$$

Ex. 1. If the curve be a parabola; $y = ax^n$;

$$\therefore dy = nax^{n-1}dx,$$
$$d^2y = n.(n-1)ax^{n-2}dx^2,$$
$$d^3y = n.(n-1).(n-2).ax^{n-3}dx^3,$$
$$\text{and } dz = dx\sqrt{1+n^2a^2x^{2n-2}};$$

$$\therefore \frac{d^3y\,dz}{2d^2y^2} = \frac{n.(n-1).(n-2).a x^{n-3} dx^4 . \sqrt{1+n^2a^2x^{2n-2}}}{2.n^2.(n-1)^2.a^2x^{2n-4}dx^4}$$

$$= \frac{(n-2).\sqrt{1+n^2a^2x^{2n-2}}}{2.n.(n-1).a x^{n-1}}.$$

Ex. 2. Let the curve be a logarithmic spiral, and force $\propto y^n$;

$$\therefore\ \gamma = y,$$

$$\text{and } d\phi = nay^{n-1}dy;$$

$$\therefore \frac{\gamma d\phi + \phi d\gamma + 2\phi dy}{2\phi dz} = \frac{(n+3)ay^n dy}{2\phi dz} = \frac{n+3}{2}.\frac{b}{c},$$

since $dy : dz$ in a given ratio.

Hence the centripetal force is to the retarding force in a constant ratio,

and $\therefore$ the retarding force $\propto y^n$.

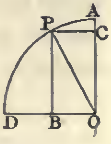
Cor. 1. Since $v^2 \propto \phi\gamma \propto y^{n+1}$;

$$\therefore \text{ density } \propto \frac{R}{v^2} \propto \frac{y^n}{y^{n+1}} \propto \frac{1}{y}.$$

Hence if density $\propto \dfrac{1}{\text{rad. vector}}$, the body may describe the logarithmic spiral whatever be the value of n.

Cor. 2. If $n = -3$, or the force $\propto \dfrac{1}{y^3}$, the resistance $= 0$.

Ex. 3. Let the curve be a quadrant of a circle, and the force act in parallel lines,



$$\gamma = PB = y;$$

$$\therefore\ d\gamma = dy,$$

$$\text{and } \frac{d\gamma + 2dy}{2dz} = \frac{3dy}{2dz} = \frac{3PC}{2PO},$$

$$\text{and centripetal force : } \phi' :: 1 : -\frac{3PC}{2PO}.$$

Cor. 1. When rad : sin AP :: 3 : 2, the retarding force is equal to the force of gravity;

and at D the retarding force : gravity :: 3 : 2.

Cor. 2. The velocity $\propto \sqrt{PB}$,

and retarding force $\propto PC$;

$$\therefore \text{ the density } \propto \frac{PC}{PB} \propto \frac{\sin AP}{\cos AP} \propto \tan. AP.$$

* * *

SECT. VI.

390. IF a stream of water impinge perpendicularly on the float-boards of a vertical wheel, and escape from it the instant it has made its impact; the force of the impinging water is proportional to the square of the difference of the velocities of the stream and the wheel.

For (337.) in general, the force $\propto A.(V \pm v)^2$

$$\propto (V - v)^2.$$

391. COR. Hence $F = aA.(V-v)^2$; where a is a constant coefficient to be determined.

392. COR. 2. The float-boards of a horizontal wheel being inclined to the horizon so as to receive the direct impact of a stream whose inclination to the horizon $= \theta$;

$$F = aA.(V - v.\cos\theta)^2.$$

For the velocity v being resolved into two; one in the direction of the stream $= v.\cos\theta$, and the other perpendicular to it and $= v.\sin\theta$, the former only will be effective in diminishing the impact. Whence the relative velocity $= V - v.\cos\theta$,

$$\text{and } \therefore\ F = aA.(V - v.\cos\theta)^2.$$

393. If a weight W be raised vertically by a string winding round the axis of a vertical wheel, the area of whose float-boards is given; to determine the velocity of the circumference of the wheel, the water being supposed to escape immediately after impact.

Let $V =$ the velocity with which the water impinges on the float-boards,

$v =$ the velocity of the circumference.

R and r = the radii of the wheel and axle, and F = the force of the water to turn the wheel when its velocity $= v$.

Then (391) $F = aA\,.\,(V - v)^2$,

$$\text{and } \therefore\ v = V - \sqrt{\frac{F}{aA}}.$$

Now the acceleration of the wheel ceases, when the force of the water to turn it, is equal to the weight which opposes it; that is, when $F\,.\,R = W\,.\,r$,

$$\text{or } F = W\,.\,\frac{r}{R};$$

$$\therefore\ v = V - \sqrt{\frac{Wr}{aAR}},$$

the velocity of the wheel when the acceleration ceases.

394. On the same supposition, to determine the weight W, so that when the wheel has acquired its uniform velocity, the momentum of W may be a max.

The uniform velocity of the ascending weight $= v\,.\,\frac{r}{R}$;

$$\therefore \text{ the momentum} = \frac{rW}{R}\,.\left(V - \sqrt{\frac{Wr}{aAR}}\right);$$

$$\text{whence } WV - W^{\frac{3}{2}}\,.\sqrt{\frac{r}{aAR}} = \text{max.}$$

$$\text{and } V\,dW - \tfrac{3}{2}\,dW\,.\,W^{\frac{1}{2}}\sqrt{\frac{r}{aAR}};$$

$$\therefore\ W = \tfrac{4}{9}\,V^2\,.\,\frac{aAR}{r}.$$

395. Cor. 1. If W' = the weight of a column of water which is equal to the force of impact,

$$W' = aAV^2;$$

$$\text{whence } W = \tfrac{4}{9}\,.\,\frac{R}{r}\,.\,W'.$$

396. Cor. 2. If W'' be a weight suspended from the axle, which would exactly balance the force of the water acting at the circumference; then

$$W'' = W' \cdot \frac{R}{r};$$

and therefore when the momentum of the ascending weight is a max. it is $= \frac{4}{9} W''$.

397. Cor. 3. To determine the velocity of the wheel when the momentum generated in the weight ascending uniformly is a max.

$$(393.) \quad v = V - \sqrt{\frac{W \cdot r}{a A R}}$$

$$= V - \tfrac{2}{3} V = \tfrac{1}{3} V \quad (394.).$$

398. Cor. 4. In the same manner it may be shewn, in the case of the horizontal wheel, that $v = \frac{1}{3} \cdot \frac{V}{\cos \theta}$.

399. Cor. 5. Since the uniform velocity of the ascending weight is $\frac{rV}{3R}$,

and (394.) the weight moved is $\frac{4}{9} \cdot V^2 \cdot \frac{aAR}{r}$;

$\therefore$ the momentum generated $= \frac{4}{27} \cdot aAV^3$.

400. Cor. 6. Hence also the momentum generated $= \frac{4}{27} \cdot W'V$.

401. Given the weight W; to determine the ratio of the radii of the wheel and axle, so that the uniform velocity of the ascending weight may be a max.

This uniform velocity $= \frac{r}{R} \cdot \left(V - \sqrt{\frac{Wr}{aAR}}\right)$;

$$\therefore \frac{V}{R} - \sqrt{\frac{Wr}{aA}} \cdot \frac{1}{R^{\frac{3}{2}}} = \text{max.}$$

$$\text{and } -\frac{V\,dR}{R^2}+\tfrac{3}{2}\cdot\sqrt{\frac{Wr}{aA}}\cdot\frac{dR}{R^{\frac{5}{2}}}=0;$$

$$\therefore\ \sqrt{R}=\tfrac{3}{2}\cdot\sqrt{\frac{Wr}{aAV^2}};$$

$$\therefore\ R:r::9W:4aAV^2$$
$$::9W:4W'.$$

402. If all the water which passes a vertical wheel be supposed to strike it; the force of the stream will vary as the relative velocity.

For the number of particles impelling the wheel in a given time will be invariable, whatever be the velocity of the wheel.

403. On this supposition, to determine the velocity of the wheel.

$$F=aA.(V-v);$$

$$\therefore\ v=V-\frac{F}{aA}=V-\frac{Wr}{aAR}.$$

404. Cor. 1. To determine W, so that when the wheel has acquired its uniform velocity, the momentum of W may be a max.

The uniform vel. of the circumference of the wheel $=V-\dfrac{Wr}{aAR}$;

$\therefore$ the uniform velocity of the ascending weight $=\dfrac{r}{R}\cdot\left(V-\dfrac{Wr}{aAR}\right)$,

$$\text{and its momentum }=\frac{r}{R}\cdot\left(VW-\frac{W^2r}{aAR}\right);$$

$$\therefore\ VW-\frac{W^2r}{aAR}=\text{max.}$$

$$\text{and } V\,dW-\frac{2r\,W\,dW}{aAR}=0,$$

$$\text{or } W=\frac{aAVR}{2r}.$$

405. Cor. 2. In this case $W' = aAV$,

$$\text{and} \therefore\ W = \tfrac{1}{2} \cdot \frac{R}{r} \cdot W'.$$

406. Cor. 3. To determine the velocity of the wheel, when the momentum generated in the weight ascending uniformly is a max.

$$(403),\quad v = V - \frac{Wr}{aAR};$$

$$\text{and}\quad (404),\quad W = \frac{aAVR}{2r};$$

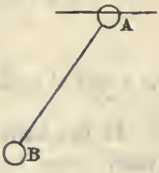
$$\therefore\ v = V - \tfrac{1}{2}V = \tfrac{1}{2}V.$$

407. Cor. 4. The uniform velocity of the ascending weight being $= \frac{1}{2} V \cdot \frac{r}{R}$,

$\therefore$ (404) the greatest momentum generated in it

$$= \tfrac{1}{2} V \cdot \frac{r}{R} \cdot \frac{aAVR}{2r} = \tfrac{1}{4} aAV^2.$$

408. Cor. 4. This momentum also $= \frac{1}{4} W'V$.



409. The *Compound Float* consists of two balls A and B joined by a thread, the one of less and the other of greater specific gravity than water; with their weights so managed, that when the instrument is thrown into the water, the upper ball remains nearly immersed. Left to the action of the stream, it acquires an equable motion, when its velocity is observed, and at the same time by means of a *simple float*, the velocity of the surface of the stream is measured. Hence (411) the velocity at B may be determined; and by lengthening the string, the velocity at different depths below the surface ascertained.

410. When the motion of the float becomes equable, to determine its velocity (v).

Let α and β be the diameters of the balls,
V and V' the velocity of the water at A and B.

The force which accelerates $A \propto \alpha^2 . (V-v)^2 = a\alpha^2 . (V-v)^2$,
and the force which retards $B = a\beta^2 . (v-V')^2$.

But as the motion, by the supposition is equable,

$$a\alpha^2 . (V-v)^2 = a\beta^2 . (v-V')^2,$$

$$\text{or} \quad \alpha . (V-v) = \beta . (v-V');$$

$$\text{whence} \quad v = \frac{\alpha V + \beta V'}{\alpha + \beta}.$$

411. Cor. 1. The velocities V and v being known from observation, V' may be ascertained.

412. Cor. 2. If the balls are equal in magnitude, the velocity of the float is an arithmetic mean between the velocities of the water at A and B.

413. Cor. 3. In order that A may be nearly immersed, the specific gravity of the float must be nearly = that of water. Let s and s' be the specific gravities of A and B.

$$\text{The specific gravity of the float} = \frac{s\alpha^3 + s'\beta^3}{\alpha^3 + \beta^3};$$

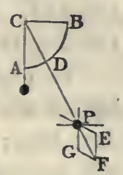
and this must be = 1 nearly,

or s and s' must be so taken, that $s\alpha^3 + s'\beta^3 = \alpha^3 + \beta^3$ nearly.

If the balls be equal in magnitude, the specific gravity of water must be an arithmetic mean between their specific gravities.

414. The *Hydraulic Quadrant* consists of a graduated vertical quadrant having two threads moving round its centre; by the shorter of which, a small weight is suspended in the air, and the longer, sustaining a weight of greater specific gravity than water, is carried by the action of a stream, whose velocity is required, from the vertical. The angle contained by the threads is measured by the intercepted arc of the quadrant.

415. Given the angle between the threads; to determine the velocity of the stream.



Let P be the weight in the water; and draw the vertical PG to represent it. Let the angle $ACD = \theta$, and the angle which the stream makes with the vertical $= \alpha$.

Since the impulse of the fluid $\propto v^2$, let it $= Qv^2$, and take PE to represent it. Complete the parallelogram EG; its diagonal PF will be in the direction of the string, and $PFG = \alpha - \theta$.

$$\text{Also } PG : PE :: \sin PFG : \sin GPF,$$

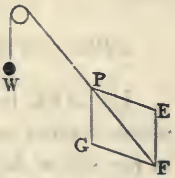
$$\text{or } P : Qv^2 :: \sin(\alpha - \theta) : \sin\theta;$$

$$\therefore\ v^2 = \frac{P}{Q} \cdot \frac{\sin\theta}{\sin(\alpha - \theta)}.$$

416. Cor. If the stream be horizontal,

$$v^2 = \frac{P}{Q} \cdot \frac{\sin\theta}{\cos\theta} = \frac{P}{Q} \cdot \tan\theta.$$

417. If the string passes over a pulley, and P be balanced by a weight W attached to its other extremity; given W, to determine the velocity of the stream.



$$PF^2 = PG^2 + GF^2 + 2PG \times GF \times \cos\alpha;$$

$$\therefore\ W^2 = P^2 + Q^2v^4 + 2PQv^2 . \cos\alpha;$$

$$\therefore\ Q^2v^4 + 2P . \cos\alpha . Qv^2 + P^2 . \cos^2\alpha$$

$$= P^2 . \cos^2\alpha - P^2 + W^2$$

$$= W^2 - P^2 . \sin^2\alpha;$$

$$\therefore\ Qv^2 = -P . \cos\alpha + \sqrt{W^2 - P^2 . \sin^2\alpha};$$

$$\text{and } v = \frac{1}{\sqrt{Q}} \cdot \sqrt{-P . \cos\alpha + \sqrt{W^2 - P^2 . \sin^2\alpha}}.$$

418. Cor. If the stream be horizontal, $\cos\alpha = 0$;

$$\therefore\ W^2 = P^2 + Q^2v^4,$$

$$\text{and } v = \frac{1}{\sqrt{Q}} \cdot \sqrt[4]{W^2 - P^2}.$$

419. The *Screw of Archimedes* consists of a tube wound spirally round a cylinder, to the surface of which it is attached. The cylinder is moveable about its axis, which is inclined to the horizon, and the ends of the tube are open.

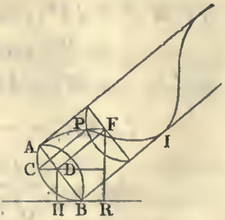
To explain the action of the screw; suppose G to be the highest and P the lowest point in the arc $AGPg$ (see fig. to Art. 433); then a ball placed in the tube between G and P will descend to P by its gravity. Now suppose the cylinder turned round its axis so as to make P ascend and the point of the spiral next to P on the right descend, then the ball descending by its gravity will move to that point or towards the upper extremity of the tube, and by the continual rotation of the cylinder will at last escape at the upper orifice. If the lower end be immersed in water, the water will rise at first in the screw to a level with the surface, and being afterwards forced to the upper orifice, by the action of the screw in the manner above described, will be there discharged.

The *base* of the cylinder not being entirely immersed; at each revolution round the axis, whilst the lower orifice is out of the water, a portion of air is introduced which separates the strata and causes an intermittent discharge from the upper orifice.

420. To determine the equation of the screw.

Let ABC be the base of the cylinder whose radius $= 1$. From any point P in the screw draw PC perpendicular to the base, and from C draw CD perpendicular to its diameter.

Let $AD = x$, $DC = y$, $CP = z$, the arc $AC = \theta$, and the arc of the screw $AP = s$.



Then by the property of the circle $y^2 = 2x - x^2$; and by the construction of the screw, if $a =$ the constant inclination of the screw to the plane of the base of the cylinder,

$$dz = d\theta \,.\, \tan \alpha;$$

$$\therefore\ z = \theta \,.\, \tan.\ \alpha.$$

These two equations give the double curvature of the screw.

421. Cor. By the property of the screw $s = \theta \,.\, \sec \alpha$; whence any arc of the screw has a constant ratio to the arc of the periphery of the base which is under it.

Let $\theta = 2\pi$, the length of the screw $= 2\pi \,.\, \sec \alpha$.

422. To determine the altitude of any point P above the horizon.

Through B draw the horizontal plane HBR; and let the angle ABH, which measures the elevation of the base above the horizon, or the declination of its axis from the vertical $= E$. Through P draw a circle parallel to the base; and complete the rectangle $PCDF$. From D and F let fall the perpendiculars DH, FR; and draw DS parallel to the horizon.

The angle $DFS = ABH = E$;

$$\text{and } DH = DB \,.\, \sin E = (2 - x) \,.\, \sin E;$$

$$\therefore\ FR = FS + DH = FD \,.\, \cos E + DB \,.\, \sin E$$

$$= z \,.\, \cos E + (2 - x) \,.\, \sin E,$$

the height required.

423. Cor. 1. Since $z = \theta \,.\, \tan \alpha$,

$$\text{and } AD = \text{versin } AC,$$

$$\text{or } \quad x = (1 - \cos \theta);$$

$\therefore$ the altitude above the horizon may be expressed by

$$\theta \,.\, \tan \alpha : \cos E + (1 + \cos \theta) \,.\, \sin E.$$

424. Cor. 2. To determine the points where the height of the spiral above the horizon is a maximum or a minimum.

$$d\theta \,.\, \tan \alpha \cos E - d\theta \,.\, \sin \theta \,.\, \sin E = 0;$$

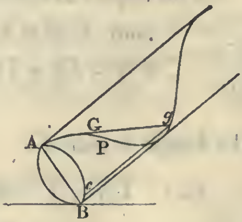
$$\therefore\ \sin \theta = \frac{\tan \alpha \,.\, \cos E}{\sin E} = \frac{\tan \alpha}{\tan E}.$$

If then θ be an arc of the circle ACB whose sine is $\frac{\tan a}{\tan E}$, the ordinate to the spiral drawn from the extremity of that arc will determine the point in the spiral where the altitude is a max.; and the ordinate drawn from the extremity of the arc $(180 - \theta)$ will give the min.

425. Cor. 3. If the screw were raised so that a became $= E$; θ would be a quadrant, and the point of the spiral thus determined would be the bisection of AI; which would not indicate either max. or min.; but merely the point of contrary flexure of that arc of the spiral which is in one half concave, and in the other convex to the base.

426. Cor. 4. If the screw were raised farther, so that a be greater than E, both values would be imaginary; and for the whole spiral, the altitude of its points above the horizon would proceed continually increasing.

427. To determine the *Hydrophorous* arc.



Through the point G corresponding to the greatest altitude above the horizon draw the horizontal plane Gg. The arc GPg will be the hydrophorous arc.

It is evident, that in a revolution of the cylinder, only the portion GPg of the spiral tube can be filled with water; for if more entered, rising above the point G it would fall back, and flow out of the lower orifice.

428. Cor. To measure the length of the hydrophorous arc, the point g must be determined, which is on a level with G.

Let the circular arc ABc corresponding to it $= \pi + X$, the arc AC corresponding to G being $= m$; the height (423) of g above the horizon

$$= (\pi + X) . \tan. a . \cos E + (1 - \cos X) . \sin E,$$

since $\cos(\pi + X) = -\cos X$,

and the height of $G = m \,.\, \tan a \,.\, \cos E + (1 + \cos m) \,.\, \sin E$.

In order ∴ to determine X, there arises the equation

$$\cos X - X \,.\, \frac{\tan a}{\tan E} = (\pi - m) \,.\, \frac{\tan a}{\tan E} - \cos m,$$

or (424) $\cos X - X \,.\, \sin m = (\pi - m) \,.\, \sin m - \cos m$;

from which equation X being determined, CBc, which is $\pi + X - m$, is also known; and therefore also the length of the hydrophorous arc $= (\pi + X - m) \,.\, \sec a$.

429. To determine the quantity discharged by the screw.

This will be determined by the length of the hydrophorous arc: since at each revolution as much will be discharged from the upper orifice as one hydrophorous arc contains. If therefore its length (428) be measured, and multiplied by a section of the tube, the quantity discharged in each revolution will be ascertained.

430. Cor. 1. If a be greater than E (426), m is imaginary, and there is no hydrophorous arc. Hence therefore it is an essential condition that the angle of declination from the vertical should be greater than the angle of inclination of the spiral tube to the base.

431. Cor. 2. The less a is compared with E, so much the less will be the arc m, and so much the more extended the hydrophorous arc. The spiral tube should therefore wind round the cylinder in as narrow a compass as possible; and the cylinder should be inclined to the horizon as much as the altitude will permit, to which the water is to be carried, and the strength of the machine.

432. Cor. 3. In order that the screw may supply as great a discharge as it is capable of, the base of the cylinder must be so far immersed in the water that its surface may touch the beginning of the circular arc which subtends the hydrophorous arc. To immerse it deeper would be useless,

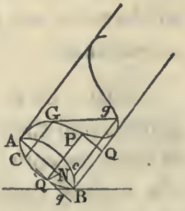
since only GPg is filled with water. If elevated higher, the whole arc GPg would not be filled.

The arc therefore rising out of the water should $= 2m$;

$$\text{and its chord} = 2 \,.\, \sin m = 2 \,.\, \frac{\tan \alpha}{\tan E}.$$

433. To determine the force necessary to keep the screw when full in equilibrio.

Let $W =$ the weight of the water in GPg. This weight, which may be supposed to be collected in the lowest point P, tends vertically downwards. Let it be resolved into two, one in the direction PQ' the side of the cylinder, and the other PQ perpendicular to it; the latter of which is the only effective force, and $= W \,.\, \sin E$.



Draw $Q'q$ parallel to PQ. Then it is evident that the distance from the axis of the cylinder at which the force PQ is applied, is equal to the ordinate $Q'N$. But $Q'N = \sin AQ' = \sin(\pi - m) = \sin m$; $\therefore$ the momentum of the water GPg to turn the cylinder in a contrary direction is $=$

$$W \,.\, \sin m \,.\, \sin E = W \,.\, \tan \alpha \,.\, \cos E.$$

And since when the screw is full there are as many hydrophorous arcs as spires, and the number of spires is equal to the length of the axis divided by $2\pi \,.\, \tan \alpha$; if $L =$ the length of the axis,

$$\text{the sum of the momenta} = \frac{L}{2\pi \,.\, \tan \alpha} \times W \,.\, \tan \alpha \cos E$$

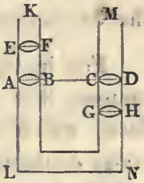
$$= \frac{LW}{2\pi} \,.\, \cos E.$$

Let now $F =$ the force required, acting at the extremity of a handle whose length is l; its momentum $= Fl$;

$$\therefore \; Fl = \frac{LW}{2\pi} \, . \cos. E,$$

$$\text{and} \;\; F = \frac{LW}{2\pi l} \, . \cos E.$$

434. If water ascend and descend in the erect legs of a cylindrical canal which is every where of the same diameter; it will perform its oscillations in the same time as a pendulum whose length is half that of the oscillating column.



Let $L =$ the length of the axis of the fluid in the canal; AB, CD its surfaces at rest. Take $BF = CG$; then if the surface CD be depressed to GH, the surface AB will be elevated to EF; therefore, the excess of the water in the leg KL above that in MN will be equal to twice the weight of the column AF, or to $A \times 2FB$; if $A =$ the area of the horizontal section;

$$\therefore \; gA \times 2FB = \text{the moving force,}$$

$$\text{and} \;\; L \times A = \text{the quantity of matter moved;}$$

$$\therefore \text{ the accelerating force} = g \, . \frac{A \, . \, 2FB}{L \times A} = g \, . \frac{2FB}{L}.$$

If a pendulum be constructed whose length $l = \frac{1}{2} L$, and an arc θ be taken $= FB$, the accelerating force of the body beginning to oscillate at the distance θ from the lowest point $= g \, . \frac{\theta}{l} = g \, . \frac{2FB}{L} =$ the accelerating force on the water in the canal. And these forces vary according to the same law; therefore the time of an oscillation of the pendulum is $=$ that of the water.

435. Cor. 1. To find the actual time.

The time (T) of an oscillation : t down $\frac{1}{2} l$ :: 2π : 1;

$$\therefore \; T = 2\pi \sqrt{\frac{l}{g}} = \pi \sqrt{\frac{2L}{g}}.$$

436. Cor. 2. Since the accelerating force $\propto FB$ the space described by the water, the oscillations will be isochronous.

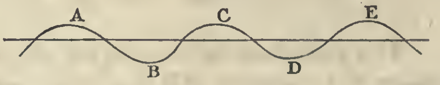
437. Cor. 3. If the oscillating column be increased or diminished, the times in which the oscillations are made will increase or decrease in the subduplicate ratio of the length of the pendulum.

438. Cor. 4. From this proposition Sir Isaac Newton inferred that the undulations of waves are of the same kind as the oscillations of water in this canal.

For in waves, the undulations are performed in such a manner that the highest parts become the lowest, and the force which depresses the eminences is always the weight of water contained in those eminences.

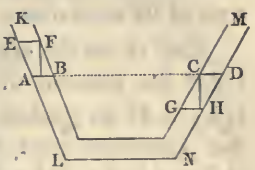
This however he considered merely an approximation.

439. Cor. 5. The undulations of waves are performed in the same time as the oscillations of a pendulum whose length is equal to the breadth of a wave.



Let $ABCDE$ represent the section of two waves. A and C will descend by the force of gravity, or a force equal to the weight of the elevated portion; and the undulations will be performed like those of water in the canal. Hence, if a pendulum be constructed whose length l is equal to half the distance between A and C, or B and D; the parts A and C will descend so as to be the lowest in the time of one oscillation of the pendulum, and in the time of another they will become the highest, or the pendulum will vibrate twice whilst A has moved to C. But a pendulum, whose length $= 4l$, will vibrate once in the same time that this oscillates twice; hence the waves perform their undulations in the same time as a pendulum whose length $= AC$ the breadth of the wave.

440. If the legs of the canal be inclined at any angles to the horizon; to determine the time of an oscillation.



Let ϕ and ϕ' be the angles of inclination, and $A =$ the area of a section perpendicular to the sides. Take $FB = DH$; then if the surface CD be depressed to GH, the surface AB will be elevated to EF, since the contents of these columns are equal. Draw FY, CX perpendicular to AB and GH;

$$\text{then } FY = FB \,.\, \sin\phi,$$
$$\text{and } CX = CG \,.\, \sin\phi' = FB \,.\, \sin\phi'.$$

Hence AB is raised perpendicularly through a space $FB \,.\, \sin\phi$, and CD depressed through $FB \,.\, \sin\phi'$;

$\therefore$ the whole moving force $= g(\sin\phi + \sin\phi') \,.\, A \,.\, FB$;
and the quantity moved $= L \times A$;

$$\therefore \text{ the accelerating force } = g \,.\, \frac{(\sin\phi + \sin\phi') \,.\, FB}{L}.$$

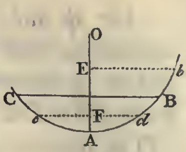
Let a pendulum be constructed whose length $= \dfrac{L}{\sin\phi + \sin\phi'}$, and an arc θ be taken $= FB$; then if a body begin to oscillate from the upper extremity of this arc, the other being the lowest point, the force at the beginning $= g \,.\, \dfrac{(\sin\phi + \sin\phi') \,.\, FB}{L}$ which is equal to the force accelerating the surface of the fluid at the beginning; therefore the times of vibration are equal, as before.

$$\text{Also } T = 2\pi \sqrt{\frac{L}{g \,.\, (\sin\phi + \sin\phi')}}.$$

441. Cor. If $\sin\phi = \sin\phi' = 1$,

$$T = 2\pi \sqrt{\frac{L}{2g}} = \pi \sqrt{\frac{2L}{g}} \text{ as before.}$$

442. Cor. If a slender column of fluid oscillates in a circular tube; to determine the velocity of the fluid, and the time of its oscillations.



Let O be the centre of the tube, and CAB the position of the fluid when at rest; cAb its position at the time (t) from the commencement of its motion. Let $AB = AC = a$, $Bb = Bd = z$, and let $A =$ the area of a section perpendicular to the axis of the tube.

The moving force is equal to the pressure of the fluid in dBb, or to the pressure of a column whose altitude is EF.

$$\text{Now } OF = \cos Ad = \cos(a - z),$$

$$\text{and } OE = \cos Ab = \cos(a + z);$$

$$\therefore EF = \cos(a - z) - \cos(a + z).$$

Also the mass moved $=$ the fluid $cAb = 2a \times A$;

$$\therefore \text{ the accelerating force} = g \cdot \frac{\cos(a - z) - \cos(a + z)}{2a}.$$

$$\text{And } v\,dv = -F\,dz = -\frac{g}{2a} \cdot \{\cos(a - z) - \cos(a + z)\} \cdot dz;$$

$$\therefore v^2 = \frac{g}{a} \cdot \{\sin(a - z) + \sin.(a + z)\} + C;$$

$$\text{when } v = 0, \text{ let } z = z';$$

$$\therefore v^2 = \frac{g}{a} \cdot \{\sin(a - z) + \sin(a + z) - \sin(a - z') - \sin(a + z')\}$$

$$= \frac{2g}{a} \cdot \{\sin a \times \cos z - \sin a \times \cos z'\}$$

$$= \frac{2g}{a} \cdot \sin a \cdot \{\cos z - \cos z'\};$$

$$\therefore v = \sqrt{\frac{2g \cdot \sin a}{a}} \cdot \sqrt{\cos z - \cos z'};$$

$$\text{whence } t = -\int \frac{dz}{v} = \sqrt{\frac{a}{2g \cdot \sin a}} \times \int \frac{dz}{\sqrt{\cos z - \cos z'}},$$

which cannot be determined generally.

If the greatest amplitude of the oscillations be small, it may be determined in the following manner:

$$\cos z = 1 - \frac{z^2}{1.2},$$

$$\cos z' = 1 - \frac{z'^2}{1.2};$$

$$\therefore \cos z - \cos z' = \tfrac{1}{2}\{z'^2 - z^2\},$$

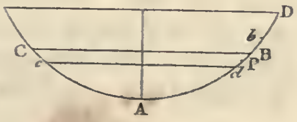
$$\text{and } t = \sqrt{\frac{a}{g.\sin a}} \times \int \frac{-dz}{\sqrt{z'^2 - z^2}}$$

$$= \sqrt{\frac{a}{g.\sin a}} \times \text{arc}\left\{\cos = \frac{z}{z'}\right\} + C, \text{ and } C = 0;$$

$$\therefore t = \sqrt{\frac{a}{g.\sin a}} \times \text{arc}\left\{\cos = \frac{z}{z'}\right\},$$

and the whole time of an oscillation $= \pi \sqrt{\dfrac{a}{g.\sin a}}$.

443. If the tube in which the fluid oscillates be a cycloid; the oscillations will be isochronous.



Let CAB be the original space occupied by the fluid, and let B be raised to b, and consequently C depressed to c; Cc being $= Bb$.

$$\left.\begin{aligned} \text{Let } AD &= a, \\ AB &= b, \end{aligned}\right\} Bb = Cc = Bd = z.$$

Taking any point P in the fluid, let $AP = x$.

Now the force at $D = g$;

and $\therefore$ the force at $P = \dfrac{gx}{a}$;

whence the pressure on dx at $P = \dfrac{g}{a}.xdx$;

$\therefore$ the whole pressure $= \dfrac{g}{2a}.x^2 + C$.

But since cA balances Ad, and the whole motion is produced by the pressure of bd;

$$\therefore \text{ the whole pressure } = \frac{g}{2a} \cdot \{Ab^2 - Ad^2\}$$

$$= \frac{g}{2a} \cdot \{(b+z)^2 - (b-z)^2\}$$

$$= \frac{g}{2a} \cdot 4bz$$

$$= \frac{2gb}{a} \cdot z = \text{the moving force};$$

and the mass moved $= 2b$;

$$\therefore \text{ the accelerating force } = \frac{g}{a} \cdot z \propto z \propto Bb,$$

whence the oscillations will be isochronous.

444. Cor. The oscillations will be performed in the same time as those of a pendulum whose length $= a$.

* * *

SECT. VII.

445. DEF. A FORCE applied at the surface of any fluid is called a *Compressing Force.*

446. DEF. An *Elastic Fluid* is one whose dimensions are diminished by increasing, and increased by diminishing the compressing force.

It is not implied in this definition, that the dimensions of an elastic fluid can be increased or diminished without limit; but that so far as human power can increase or diminish the compressing force, the dimensions of the fluid may be diminished or increased. Experiment shews that the atmosphere near the surface of the earth is compressed by the weight of the superincumbent column of air into at least the 13000th part of the space it would occupy *in vacuo.* And if it were rarefied and condensed to the greatest degree that has yet been effected, the ratio of the spaces occupied by the same quantity in its states of greatest rarefaction and condensation would be as 550000 : 1.

447. If a fluid consist of particles which repel each other with forces varying inversely as the n^{th} power of the distances between their centres; the compressing force will vary as the $(n+2)^{\text{th}}$ power of that distance.

Let a vertical rectangular parallelopiped of the fluid be intersected by a plane parallel to the horizon. Then the fluid being at rest, the pressures upwards and downwards against the plane are equal. Now the pressure upwards, that is, the action of the square surface contiguous to the plane, varies as the number of particles in the surface and the force of each jointly: and this number varies as the number of particles in a line and the number of lines in the surface jointly. But each of these varying inversely as (d) the distance between the centres of two contiguous particles, the action upwards $\propto \frac{1}{d^2} \times \frac{1}{d^n} \propto \frac{1}{d^{n+2}}$. And the pressure downwards is C the compressing force;

$$\therefore\ C \propto \frac{1}{d^{n+2}}.$$

448. Cor. 1. This fluid will be elastic if $n+2$ be positive.

For suppose the compressing force increased; then d^{n+2}, and therefore d is diminished, that is, the dimensions of the fluid are diminished. And if the compressing force be diminished, the dimensions of the fluid will be increased; therefore the fluid has the properties of an elastic fluid (446).

449. Cor. 2. The fluid will be elastic if the particles repel each other with forces varying *inversely* as *any* root or power of the distance between their centres; or *directly* as any root or power less than the direct duplicate.

For in the first case n, and therefore $n+2$ is positive; and in the second, n is a negative number less than 2, and therefore $n+2$ is positive, or the fluid will be elastic.

450. If the particles of an elastic fluid at rest be equally affected by gravity, the fluid will be of an uniform density.

Let A, B, C be three contiguous particles in the same horizontal plane, and therefore equally affected by gravity; then is B equally distant from A and C. For if possible, let it be nearer to A than to C; in which case, if the repulsive force of the particles $\propto \frac{1}{d^n}$, B will be more repelled by A than by C; but if the repulsive force $\propto d^n$ where n is less than 2, B will be less repelled by A than by C; whence motion will ensue, which is contrary to the supposition. A, B and C therefore must be equally distant. And as this may be proved of any other three contiguous particles under the same circumstances, the density of the fluid must be uniform.

451. If the law be given according to which the particles repel each other; to determine to what root or power of the density the compressing force is proportional.

Let a vertical rectangular parallelopiped of the fluid be intersected by any number of planes parallel to the horizon, and therefore to each other; the distance between each two being equal to a side of the base. Then the fluid between any two contiguous planes occupies a cubical space; and since $Q \propto M \times D$, and in this case M is given, $Q \propto D$. But the quantity contained in a cubical space is as the number of particles in a line, the number of lines in a surface, and the number of surfaces in the cube; for any two being given, the number varies as the other; and each of these varying as $\frac{1}{d}$; $\therefore Q \propto \frac{1}{d^3}$, and $\therefore D \propto \frac{1}{d^3}$.

$$\text{Hence } D^{\frac{n+2}{3}} \propto \frac{1}{d^{n+2}} \propto C \quad (447).$$

If therefore the law by which the particles repel each other, be known, that is, if n be known, the variation of the compressing force in terms of the density may be determined.

452. The atmosphere is found by experiment to be affected by the force of gravity.

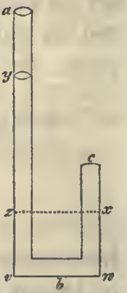
For a vessel when exhausted weighs less than when filled with air.

453. Cor. If then the atmosphere be supposed to be divided into an indefinite number of laminæ perpendicular to the direction of gravity, it is manifest that the lower will be pressed by the weight of those above: and this compression will increase *cæt. par.* in proportion as the compressed lamina is lower in the atmosphere.

454. The weight of water is to that of air as 800 : 1 nearly. It must however be observed, that the weight of air is different at different times. The ratio of 885 : 1 has been deduced, the barometer standing at $29\frac{1}{2}$ inches; and that of 850 : 1, the barometer standing at 30 inches. The ratio of 836 : 1 has also been deduced, the barometer standing at 29.27 inches, and Fahrenheit's thermometer at 53^0; which reduced to 30 of the barometer and 55^0 of the thermometer is as 833 : 1.

455. Air is an elastic fluid, the density of which at a given temperature is proportional to the compressing force.

1. Air is a fluid; since it yields to the smallest inequality of pressure, and its parts are easily moved amongst themselves.



2. It is elastic. Let a cylindrical glass tube *abc* consisting of two parallel branches connected by a third at right angles to them be so placed that its parallel branches may be perpendicular to the horizon. Into the open end let a quantity of mercury be poured sufficient to fill the base *vw*; and the end *c* be hermetically sealed; *cw* having a graduation expressing equal parts of its capacity. The air in *cw* is in its natural state; and the force compressing it is the weight of a column of atmosphere of the same base and altitude equal to that of the atmosphere; or (as will be shewn Art. 541.) the weight of a column of mercury whose altitude H is the height of the barometer at the time of the experiment. If now mercury be poured into *ab*, it will rise also in *cw*, suppose to x. The compressing force in this case is the weight of the mercurial column $(zy+H)$, since if a horizontal plane xz intersect both branches of the tube contiguous to the surface of the mercury at x, the columns vz and wx will balance each other (83). Then since $D \propto \frac{1}{M}$,

the density in cw : the density in cx :: cx : cw;

hence the dimensions are diminished by increasing the compressing force; and, by reversing the experiment, it will appear that they are increased by diminishing it; or the air is an elastic fluid.

3. Also from the graduation it appears that

$$cx \, . \, (zy + H) = cw \, . \, H,$$

$$\text{or } cx : cw :: H : zy + H;$$

but the compressing forces are as $H : zy + H$;

$\therefore$ the densities are as the compressing forces.

By reversing the experiment the same may be proved when the air is rarefied.

456. Care must be taken that the air inclosed in the tube wc should be of the temperature of the external air, and consequently that the space occupied should only vary from the variation of the compressing force. Without such precaution, heat and cold not acting equally on the two airs will cause a difference in the results; and it will be difficult to separate the effects produced by them from those of the compressing force. The included air must also be free from vapours, and the tube dried from any humidity which may attach itself to the surface of the glass: for if there be any vapour, its pressure will not be according to the same law. And the experiments should not be made rapidly in succession.

457. Cor. 1. Since the elastic force is equal to the force of compression*; the different elastic forces of the same

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* That the elastic force is equal to the force of compression is evident from the principle that action and re-action are equal; for admitting the air to be in a state of equilibrium, there must necessarily exist a force in any given quantity of air to resist and balance the external pressure. The same is also evident from experiment. For let one end of an open tube be immersed in mercury contained in a closed unexhausted vessel: if the tube and vessel be placed under the

mass of air when occupying different spaces are inversely as those spaces.

458. Cor. 2. If $P =$ the pressure of the air in its natural state in *wc*, and p' any other pressure, S and S' the spaces occupied in the two cases, then

$$S : S' :: p' : p;$$

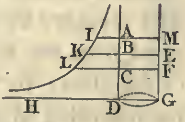
$$\text{and } \therefore \; S' = S \,.\, \frac{p}{p'};$$

whence if the first pressure be known and the space originally occupied by any portion of air; the space it will occupy, or the magnitude to which it will be reduced by any new pressure p', may be determined.

Also if p'' be any other pressure, and S'' the corresponding space occupied,

$$S'' = S \times \frac{p}{p''} = S' \times \frac{p'}{p''};$$

whence S'' may be determined from S', independently of the initial pressure.



459. Cor. 3. Since the space in which any quantity of air is confined varies inversely as the compressing force, the forces which confine a quantity of air in the cylindrical spaces *AG*, *BG*, *CG*, are inversely as those spaces, or inversely as the altitudes *AD*, *BD*, *CD*. Hence if with *AD* and *DH*, which is perpendicular to it, as asymptotes, the hyperbola *IKL* be described, and the ordinates *AI*, *BK*, *CL*

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the receiver of an air-pump, on exhausting the air from the receiver, the mercury rises in the tube nearly to the standard altitude.

Hence in any closed unexhausted vessel which removes the incumbent pressure of the external air from substances placed within it, the substances so placed sustain a pressure from the air's elastic force equal to the weight they would have to sustain from the pressure of the atmosphere.

be drawn; the forces which compress the air in AG, BG, CG, will be directly as the ordinates AI, BK, CL.

460. Cor. 4. Particles of air repel each other with forces varying inversely as the distances between their centres.

When the repelling force of the particles $\propto \frac{1}{d^n}$, (451) the compressing force $C \propto D^{\frac{n+2}{3}}$. But in this case $C \propto D$;

$$\therefore \frac{n+2}{3} = 1, \text{ and } n = 1;$$

$$\text{or the repelling force} \propto \frac{1}{d}.$$

461. Cor. 5. Since water is but in a very small degree compressible, the particles must be kept asunder by some repulsive force, whilst D remains nearly constant. And since $C \propto D^{\frac{n+2}{3}}$; $D \propto C^{\frac{3}{n+2}}$; in order therefore that $C^{\frac{3}{n+2}}$ may be nearly invariable, n must be a very great number, or the repulsive force of the particles of water must be inversely as a high power of their distance.

462. Cor. 6. If the compressing force $\propto D^{\frac{n+2}{3}}$, the repulsive force will $\propto \frac{1}{d^n}$.

For since $Q \propto MD$, if the same supposition be made as in Art. 451, $Q \propto D$, and $\therefore D \propto \frac{1}{d^3}$,

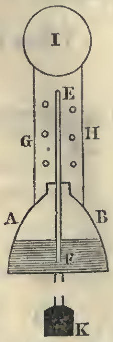
$$\text{and } D^{\frac{n+2}{3}} \propto \frac{1}{d^{n+2}};$$

$\therefore$ the compressing force $\propto \frac{1}{d^n} \times \frac{1}{d^2} \propto \frac{1}{d^n} \times$ the number of particles in the surface.

But $C \propto$ the number of particles and force of each;

$$\therefore \text{ the force of each} \propto \frac{1}{d^n}.$$

463. The *Sea-Gage* is an instrument for measuring the depth of the sea, when it cannot conveniently be done with a fathom-line. It consists of a gage-bottle AB into which is fixed the gage-tube EF, the upper end of which is hermetically sealed, and the lower immersed in mercury. This is enclosed in a tube GH, pierced with holes so as to admit the water into the bottle AB, and having on its top a large empty ball or full-blown bladder. To the bottom of AB is fixed by a spring a large weight K.



When K is attached, it causes the instrument to sink, till striking against the bottom of the sea, it becomes detached from AB, when I buoys up the gage to the surface. In the descent, the water pressing on the surface of the mercury causes it to rise in the gage-tube EF. If therefore on the surface of the mercury a small quantity of viscid matter be placed, which will adhere to the tube; the height to which it has been forced by the pressure at the bottom of the sea will be shewn; and therefore the space into which the air in the tube EF has been compressed, and consequently the depth of the water, which is reciprocally proportional to the space occupied by the air, may be ascertained.

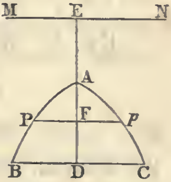
464. The *Diving-Bell* is a machine for the purpose of conveying persons below the surface of the sea, and enabling them to respire.

It may be of any figure; but was usually that of a bell or a truncated cone with the larger base downwards; in which the water ascends to a less altitude than if it had been a hollow cylinder. But the form of a parallelopiped has been found more convenient and safe for the workmen; and in consequence has been adopted on the suggestion of Mr. Smeaton.

It is poised with lead or heavy materials, and suspended in such a manner that it may descend with its open base downwards and parallel to the horizon, filled with common air. During the descent, the dimensions of the included air become contracted by the increasing weight of the water which compresses it. At the depth of about 34 feet the bell is half full of water, the pressure of the incumbent water being then equal to that of the atmosphere: and supposing its figure regular, the capacity of the whole and the part occupied by the air in any case being compared, the altitude to which the water rises will be ascertained.

When the air within the bell becomes unfit for respiration, an additional supply is introduced by means of a leathern tube through which it is forced from above by a forcing-pump. In some cases the tube communicates with a barrel which contains fresh air and is let down near the bell.

465. Having given the form of the Diving-Bell and the depth it has descended; to determine how high the water will have risen.



Let MN be the surface of the water, BAC the bell, and Pp the surface of the water in it. Let $AF=x$, $FP=y$, $EA=h$, and $AD=b$. Then if $a=$ the altitude of the mercurial barometer, $14a$ will $=$ that of the water-barometer, or the altitude of a column of water which would produce the same pressure as the atmosphere. Let M and m represent the capacities of ABC and APp respectively. Then when the air occupies the space M, it is in its natural state, and therefore supports a column of water whose height $= 14a$; and when it occupies the space m, it sustains the pressure of a column of water whose height $= 14a + EF = 14a + h + x$. But since the density varies inversely as the space occupied by the same quantity,

$$14a : 14a + h + x :: m : M,$$

from which proportion x, and $\therefore$ FD may be determined.

Ex. 1. Suppose the bell to be a paraboloid, and the equation to the generating parabola to be $y^2 = 4cx$.

$$M = 2c\pi b^2, \text{ and } m = 2c\pi x^2;$$

$$\therefore\ 14a : 14a + h + x :: x^2 : b^2,$$

$$\text{or } x^3 + (14a + h) . x^2 = 14ab^2,$$

from the solution of which equation x may be found.

Ex. 2. If the bell be a cylinder or prism with its axis vertical,

$$M : m :: b : x;$$

$$\text{and } \therefore\ 14a : 14a + h + x :: x : b,$$

$$\text{whence } x^2 + (14a + h) . x = 14ab,$$

$$\text{and } x = \sqrt{\left(\frac{14a + h}{2}\right)^2 + 14ab} - \frac{14a + h}{2}.$$

Let $h = 45$ feet, $b = 5$, and suppose $a = 30$ inches;

$$\therefore\ \left(\frac{14a + h}{2}\right)^2 = 1600, \text{ and } 14ab = 35 \times 5 = 175;$$

$$\therefore\ x = \sqrt{1775} - 40 = 42.12 - 40 = 2.12 \text{ feet nearly.}$$

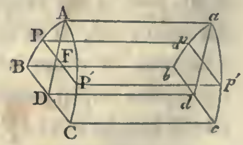
Ex. 3. If the bell be a cone, $M : m :: b^3 : x^3$;

$$\therefore\ 14a : 14a + h + x :: x^3 : b^3,$$

$$\text{whence } x^4 + (14a + h) . x^3 = 14ab^3,$$

the solution of which will give x.

Ex. 4. If the figure be generated by a parabola moving parallel to itself.



Let ABC be the generating parabola, Aa the length of the vessel, and $APpp'P'A$ the part into which the water does not enter.

Then

$$M : m :: ABC \times Aa : APP' \times Aa :: ABC : APP' :: b^{\frac{3}{2}} : x^{\frac{3}{2}},$$

$$\text{whence } 14a : 14a + h + x :: x^{\frac{3}{2}} : b^{\frac{3}{2}},$$

$$\text{and } x^{\frac{5}{2}} + (14a + h) \,.\, x^{\frac{3}{2}} = 14ab^{\frac{3}{2}},$$

from the solution of which x will be determined.

Cor. Supposing the water to have risen half way up the axis; to determine the depth of the bell.

Here $x = \frac{1}{2}b$; and $\therefore \frac{b^{\frac{5}{2}}}{\sqrt{32}} + (14a + h) \,.\, \frac{b^{\frac{3}{2}}}{\sqrt{8}} = 14ab^{\frac{3}{2}}$,

$$\text{whence } h = 14a \,.\, \{\sqrt{8} - 1\} - \tfrac{1}{2}b.$$

466. As the distance from the surface of the earth increases, the density of the air decreases.

For since the density varies as the compressing force (455), and the compressing force is the weight of the incumbent atmosphere, that force must be less at a greater than at a less altitude; whence the density must also decrease.

467. Cor. A cubic inch of the air we breathe, would be so much rarefied at the distance of 500 miles from the surface of the earth, as to fill a globe equal in diameter to the orbit of Saturn. And at a distance from the earth's surface of 100th part of its radius, the rarity of the air is greater than that to which it can be reduced in the receiver of the best air-pump. At small elevations the densities become inconsiderable.

468. Def. An *homogeneous atmosphere* is one of the same uniform density as that of the air at the earth's surface, and whose weight is equal to that of the atmosphere which surrounds the earth.

469. The altitude of an homogeneous atmosphere at any point is the same as at the earth's surface; gravity being constant.

Since the compressing force varies as the density, and also varies as the height of an homogeneous atmosphere and its density jointly;

$$\therefore\ H \times D \propto D,$$

$$\text{and } H \propto 1.$$

470. If the distances above the surface of the earth increase in arithmetical progression, the corresponding densities of the air will decrease in geometrical progression; the force of gravity being invariable.

Let a regular vertical column of atmosphere be intersected by an indefinite number of equidistant planes, parallel to each other and to the horizon, so that the density of the air may be uniform throughout the same lamina. Let the weight of the whole column $= a$, that of the column incumbent on the first lamina $= b$, on the second $= c$, &c. then are b, c, d, &c. the forces respectively compressing the first, second, third, &c. laminæ, and therefore proportional to the densities in those laminæ (455). But since the force of gravity is supposed invariable, $W \propto Q \propto D$, since M is given. Hence the densities are as $a - b$, $b - c$, $c - d$, &c.

$$\text{And } \therefore\ a - b : b - c :: b : c,$$

$$\text{whence } a : b :: b : c.$$

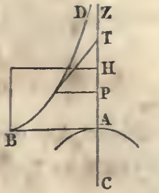
In same manner it may be shewn that $b : c :: c : d$, and so on. Hence b, c, d, &c. and therefore the densities are in a geometrical progression; which is decreasing, since a is greater than b, and therefore b than c, &c. And the thickness of each lamina being the same, the altitudes of the several laminæ above the surface increase in arithmetical progression.

471. Cor. 1. If the first density $= D$, and from each be taken its nth part, there will remain its $\left(\frac{n-1}{n}\right)$th part, and the series of densities will be

$$D,\ \frac{n-1}{n} \cdot D,\ \left(\frac{n-1}{n}\right)^2 \cdot D,\ \left(\frac{n-1}{n}\right)^3 \cdot D, \text{ \&c.}$$

the common ratio of the series being $\frac{n-1}{n}$.

472. Cor. 2. If AB be drawn perpendicular to the radius of the earth CA produced, and taken to represent the density of the air at the surface; and at any other altitudes perpendiculars be drawn, representing the corresponding densities; the locus of their extremities will be a logarithmic curve.



473. Cor. 3. The height of the homogeneous atmosphere will be equal to the subtangent of the curve.

For let AH = the height of the homogeneous atmosphere; then $AH \times AB$ will represent the whole pressure (468). But this will also be represented by the area $ABDZ$, which is $= AB \times PT$; PT being the subtangent;

$$\text{whence } AH = PT.$$

474. Cor. 4. Different altitudes above the earth's surface are as the logarithms of the densities at those altitudes.

475. To determine the figure of the atmosphere.

The atmosphere, on the supposition of the earth's being a perfect sphere, and in a state of rest, would assume the same figure; since every point in the surface of a fluid at rest must be equally distant from the common centre of attraction. But the earth and the ambient atmosphere having a diurnal motion round a common axis, the centrifugal force thence arising will cause the atmosphere to assume the figure of an oblate spheroid, the minor axis of which is the axis of rotation.

476. To determine the density of the air at any altitude, supposing the force of gravity to vary inversely as any power (n) of the distance from the earth's centre.

Let r = the radius of the earth,
z = any altitude above the surface,

δ = the density at the surface,
y = the density at the altitude z,
h = the height of an homogeneous atmosphere.

Since the density varies as the compressing force; and this varies as the quantity of matter and gravitation jointly,

$$\therefore\ 1 . \delta . h : \frac{r^n}{(r+z)^n} \times -y\,dz :: \delta : dy,$$

$$\text{and } \frac{r^n}{h} \cdot \frac{-dz}{(r+z)^n} = \frac{dy}{y};$$

$$\text{whence } \frac{r^n}{h} \times \frac{1}{(n-1) . (r+z)^{n-1}} = \text{hyp. log. } y + C.$$

And at the surface $z=0$, and $y=\delta$,

$$\therefore \frac{r^n}{(n-1) . h} \times \frac{1}{r^{n-1}} = \text{hyp log. } \delta + C;$$

$$\therefore \frac{r^n}{(n-1) . h} \cdot \left\{ \frac{1}{r^{n-1}} - \frac{1}{(r+z)^{n-1}} \right\} = \text{hyp. log. } \frac{\delta}{y}.$$

Let ϵ be the number whose hyp. log. = 1,

$$\text{then } \epsilon^{\frac{r}{(n-1) . h} \cdot \frac{(r+z)^{n-1} - r^{n-1}}{(r+z)^{n-1}}} = \frac{\delta}{y},$$

$$\text{whence } y = \delta . \frac{1}{\epsilon^{\frac{r}{(n-1) . h} \cdot \frac{(r+z)^{n-1} - r^{n-1}}{(r+z)^{n-1}}}}.$$

477. Cor. 1. If $n=2$, or the force vary inversely as the square of the distance,

$$y = \delta . \frac{1}{\epsilon^{\frac{r^2}{h} \cdot \left(\frac{1}{r} - \frac{1}{r+z}\right)}}.$$

If then $r+z$ increase in harmonic progression, $\frac{1}{r+z}$ will decrease in arithmetic progression, and y will decrease in geometric progression.

478. Cor. 2. If the force of gravity be supposed constant, or $n=0$,

$$y=\delta \cdot \frac{1}{\epsilon^{\frac{z}{h}}}.$$

479. Cor. 3. If $n=0$, and the altitudes above the surface of the earth increase in arithmetic progression, $\frac{z}{h}$ increases in arithmetic progression, and therefore the density will decrease in geometric progression.

If $z=h$, or the altitude be equal to that of an homogeneous atmosphere, $y=\frac{\delta}{\epsilon}$.

480. Cor. 4. When $n=0$, $\frac{z}{h}=\text{hyp. log.}\ \frac{\delta}{y}$, and if z' and y' be any other altitude and corresponding density,

$$\frac{z'}{h}=\text{hyp. log.}\ \frac{\delta}{y'};$$

$$\therefore\ \frac{z-z'}{h}=\text{hyp. log.}\ y'-\text{hyp. log.}\ y;$$

that is, the difference of the altitudes is proportional to the difference of the hyp. logs. of the densities at those altitudes; and therefore is proportional to the difference of the logarithms of the heights of the mercury in the barometer,

$$\text{since (549)}\quad \frac{y'}{y}=\frac{H'}{H}.$$

481. Cor. 5. If δ' = the density of mercury,

$$h\times\delta=H\times\delta';$$

$$\therefore\ h=H\cdot\frac{\delta'}{\delta}.$$

Let m = the modulus of the common system of logarithms,

$$\therefore\ z\left(=h\,.\,\text{hyp. log.}\ \frac{\delta}{y}\right)=\frac{H\delta'}{m\delta}\times\text{log.}\ \frac{H}{H''},$$

H'' being the height of the mercury in the barometer, at that altitude where the density of the air $=y$.

482. Cor. 6. If by a trigonometrical observation the altitude z has been determined, and the altitude of the mercury observed, the ratio of the density of the mercury to that of the air at the altitude z may be determined, on the same supposition; for

$$\frac{z}{H}\times\frac{m}{\text{log. }H-\text{log. }H''}=\frac{\delta'}{\delta}.$$

483. If the compressing force vary as the (m)th power of the density, and the force of gravity inversely as the (n)th power of the distance from the earth's centre: to determine the law of the density in the atmosphere.

Let $p=$ the pressure at the altitude z, the rest as in (476),

$$\text{then } \delta^m : y^m :: h : p;$$

$$\therefore\ p=\frac{hy^m}{\delta^m},\ \text{and } dp=\frac{mh}{\delta^m}\,.\,y^{m-1}dy.$$

$$\text{Now } dp\propto dQ\times\text{force of gravity}\propto\frac{r^n}{(r+z)^n}\times -ydz;$$

$$\therefore\ \frac{mhy^{m-1}dy}{\delta^m}\propto\frac{r^n}{(r+z)^n}\times -ydz,$$

$$\text{suppose } q\,.\,\frac{mhy^{m-1}dy}{\delta^m}=\frac{r^n}{(r+z)^n}\times -ydz,$$

where q is some constant quantity.

$$\text{At the surface, } q\,.\,\frac{mhdy}{\delta}=-\delta dz;$$

$$\text{and at the same time } \frac{mh}{\delta}\,.\,dy=dp=-dz;$$

$$\therefore\ q=\delta;$$

and consequently, $\frac{mh}{\delta^{m-1}} \times y^{m-2}dy = -\frac{r^n dz}{(r+z)^n}$,

whence

$$\frac{mh}{(m-1).\delta^{m-1}}.\{\delta^{m-1}-y^{m-1}\}=\frac{r}{n-1}.\frac{(r+z)^{n-1}-r^{n-1}}{(r+z)^{n-1}};$$

$$\therefore\ y^{m-1}=\delta^{m-1}-\frac{(m-1)\,\delta^{m-1}r}{(n-1).mh}+\frac{(m-1).\delta^{m-1}r^n}{(n-1).mh.(r+z)^{n-1}};$$

and

$$y=\left(\delta^{m-1}-\frac{(m-1).\delta^{m-1}r}{(n-1).mh}+\frac{(m-1).\delta^{m-1}r^n}{(n-1).mh.(r+z)^{n-1}}\right)^{\frac{1}{m-1}}.$$

484. Cor. If m and n, each $=2$,

$$y=\delta-\frac{\delta r}{2h}+\frac{\delta r^2}{2h.(r+z)}=\delta-\frac{\delta rz}{2h.(r+z)}.$$

SECT. VIII.

485. THE property which bodies possess of expanding, upon the application of heat, and of returning to their original dimensions when brought back to the same temperature, offers a simple and exact mode of measuring the degrees of heat to which they are exposed. This has been employed in the construction of *thermometers* or measures of heat.

486. Strictly speaking, any body may be used for this purpose; since all expand by heat, and contract by cold: and the quantity of this expansion is uniformly the same in the same bodies when exposed to the same temperature. But it is found to differ very considerably in different kinds of matter, under the same increase or diminution of heat. In solid bodies it is least, in liquids greater, and in elastic fluids greatest of all: and in different kinds of solids, liquids, and elastic fluids, the difference is considerable. The object in request is an instrument which shall be an exact measure of temperature, and at the same time most easily, uniformly and sensibly indicate the variations. In a solid body, *e.g.* a bar of metal, the expansions and contractions, though equable, or very nearly so, are too small to be easily observed: and if to render them more visible, recourse be had to wheels and levers, considerable errors might be expected. In air or gas, the expansion and contraction are so great as to make it extremely inconvenient to measure them, when the variation of heat is considerable. The variations of magnitude in liquids, being greater than in solids, and less than in gases, render them a more suitable standard for the measure

of heat[*]: the qualities of mercury are particularly adapted to the purpose; for it supports more heat than any other fluid before it boils, except certain oils; and it may without freezing be exposed to such a degree of cold as would render all other fluids solid, except some spirits: it is of all liquids the most easily freed from air: and the most sensible to heat and cold of any fluid, even air not excepted: it is moreover a homogeneous fluid, and every portion of it equally dilated or contracted by equal variations of heat: and in the observations usually made, the variations of its magnitude are perfectly regular and proportional to those of solids and gases under the same circumstances.

In order, however, that observations made in different places may be compared, care must be taken that the substance used be the same, that is, that the mercury be perfectly pure and separated from the metals with which it is often found combined; and that the tube in which it is inclosed be of uniform diameter.

487. DEF. The *common thermometer* is a glass tube having a bulb at the lower extremity, which with part of the tube is filled with mercury; the remaining part being a vacuum, and the top hermetically sealed.

488. To fill a thermometer.

The bore of the tube being so small that mercury cannot be poured into it, let the bulb be heated by the flame of a candle or lamp blown against it with a blow-pipe. By this means the air contained within it will expand and be expelled: if then the orifice be immersed in a vessel of mercury, and the bulb cooled, the mercury will ascend from the pressure of the external air.

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[*] Air is the most expansible fluid: but it neither receives nor parts with its heat so quickly as mercury. Alcohol expands little by heat, and when highly rectified, can bear a greater degree of cold than any other liquor hitherto employed as a measure of temperature. But it is not possible to get it always of the same strength. The expansion of oil is about fifteen times greater than that of alcohol: it sustains extreme degrees of heat and cold, but its viscidity renders it almost useless; for adhering to the sides of the tube, it cannot suddenly shew any change of temperature.

By heating the tube before the mercury is introduced, the small quantity of air and moisture which adheres to the glass in the ordinary state of the atmosphere, is expelled; and the little impurities which would disturb the motion of the mercury along the tube will also be dispersed.

If upon this operation a sufficient quantity of mercury is not introduced; let the bulb and the mercury in it be heated, and immersed again; and the operation repeated. In general, for common use the tube half filled will be sufficient.

Particular care must be taken to expel all the air: since its expansion and compressibility, which are different from those of mercury, would alter the regularity of its motion. The only accurate mode of doing this is to heat the bulb till the mercury boils. But this would expel some of the mercury, unless when expanding it rises into a bulb on the top, and when cool re-enters the tube. When the air is completely excluded, let the tube be hermetically sealed.

489. To graduate the scale of a thermometer.

If the thermometer be immerged in melting snow, the mercury will descend and stop at a certain point; from which it will not vary till the snow is entirely melted, even though there should be a variation of temperature in the atmosphere: and this point remaining the same in every experiment will shew the temperature of melting snow. In different substances the column of mercury will rest at different points, which would thus indicate their several temperatures. But these points being, in general, different in different thermometers; their position depending upon the ratio of the capacity of the bulb and tube, and the quantity of mercury introduced; the thermometers so formed would not be capable of comparison: and it becomes consequently necessary to find some other fixed temperature, the distance of which from that of melting snow may be divided into a certain number of equal parts. The magnitudes of these parts are independent of the dimensions of the thermometer, as will appear hereafter.

When the thermometer is plunged into boiling water, the mercury rises to a certain point and there remains fixed, whatever heat is applied to the vessel; provided the water be not evaporated. This therefore will serve for a second fixed point. Let the interval between these two fixed points be divided into a certain number of equal parts; and the divisions carried as far above and below them as may be wanted.

490. Thermometers thus constructed, will, when exposed to the same temperature, shew the same number of equal parts.

Let M and m be the magnitudes of the mercury in two thermometers,

R and r the interior radii of the tubes,

L and l the lengths between the two fixed points, divided into the same number (n) of equal parts.

Upon this exposure to the same temperature (t), let L' and l' be the heights above the point of melting snow; and let ϵ, ϵ' be the expansions of a particle of mercury from that point respectively to boiling water, and to the temperature t; the corresponding expansions of the mercury in the first thermometer will be $M\epsilon$, $M\epsilon'$; and in the second $m\epsilon$, $m\epsilon'$; since they are proportional to the magnitudes. But these are also measured by cylinders of mercury of known altitudes;

$$\text{whence } \left.\begin{matrix} M\epsilon = \pi R^2 L \\ \text{and } m\epsilon = \pi r^2 l \end{matrix}\right\}, \text{ also } \left\{\begin{matrix} M\epsilon' = \pi R^2 L' \\ m\epsilon' = \pi r^2 l'. \end{matrix}\right.$$

Also L being divided into n equal parts, L' will contain $\frac{nL'}{L}$ of them, and the number in the second will be $\frac{nl'}{l}$. And these will be equal; for dividing each equation by each,

$$\frac{L'}{L} = \frac{\epsilon'}{\epsilon} = \frac{l'}{l};$$

$$\text{and } \therefore \frac{nL'}{L} = \frac{nl'}{l}.$$

The thermometers therefore when exposed to the same heat, will shew the same number of degrees, and therefore may be compared.

491. Cor. Since $\frac{L'}{L} = \frac{l'}{l}$; $\therefore L' = \frac{l'}{l} . L$,

if $l' = \frac{1}{p} . l, \ \frac{2}{p} . l, \ \frac{3}{p} . l$, &c. ($p$ indicating the number of equal parts or degrees into which the interval between melting snow and boiling water is divided) one thermometer will indicate the temperatures increasing by single degrees. But the corresponding values of L' will be $\frac{1}{p} . L, \ \frac{2}{p} . L, \ \frac{3}{p} . L$, &c., that is, the divisions will be equal to each other, and proportional to the first. Hence, therefore, a thermometer made of pure mercury is possessed of the same properties as any other thermometer made of pure mercury: and their correspondence does not depend upon the law of their expansion.

492. In Fahrenheit's thermometer, which is chiefly used in England, the point of melting snow is marked 32^0, and boiling water 212^0; the distance therefore between the two fixed points is 180^0. In this $55°$ answers to temperate, 76^0 to summer heat, and 90^0 to blood heat.

In Reaumur's, the point of melting snow is marked $0°$, and boiling water $80°$, which corresponds with 212^0 of Fahrenheit. Any other degrees of temperature in the one may easily be converted into the corresponding degrees of the other. For the interval between the fixed points in Fahrenheit being 180^0, and in Reaumur 80^0, each of Fahrenheit's is $\frac{4}{9}$ of one of Reaumur's; and F being any degree of the former, and R the corresponding degree in the latter,

$$F=\frac{R\,.\,9}{4}+32,\quad\text{and}\quad R=\frac{(F-32)\,.\,4}{9}.$$

The French make use of the centesimal division, which was used by Celsius; marking their point of melting snow 0^0, and boiling water 100^0. Each degree therefore of Fahrenheit's is $\frac{5}{9}$ of one on this scale;

$$\therefore\ F=\frac{C\,.\,9}{5}+32,\quad\text{and}\quad C=\frac{(F-32)\,.\,5}{9}.$$

In Newton's thermometer of linseed oil, the point[*] of melting snow is 0^0, and of boiling water 34^0. By this he determined the degrees of fusion of a great number of substances, and found each invariable.

493. Given the degree of temperature in one thermometer; to determine generally the corresponding degree in another which is differently graduated.

Let n be the degree of temperature in the former, and t the required degree in the latter. Then since they must rise or fall together, t will be $=a+bn$, a and b being constant quantities to be determined. Now if x^0 of the former answer to N^0 of the latter, and y^0 to N'^0; which relation is known from knowing the interval in each between melting snow and boiling water;

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[*] Newton first determined the important point on which the accuracy and the value of the thermometer depends: having chosen as fixed those points at which water freezes and boils; the very points which have been determined to be the most fixed and convenient.

$$x = a + bN$$

$$y = a + bN';$$

$$\therefore\ b = \frac{y - x}{N' - N},$$

$$\text{and}\quad a = x - \frac{(y - x) . N}{N' - N} = \frac{N'x - Ny}{N' - N};$$

$$\therefore\ t = x + \frac{(y - x) . (n - N)}{N' - N}.$$

494. Cor. 1. Let $x = 0$, and $y = 100$, as in the centesimal thermometer;

$$\therefore\ t = 100 . \frac{n - N}{N' - N},$$

which is easily verified. For $N' - N$ being the number of degrees between the points of melting snow and boiling water, will correspond to 100 in the centesimal thermometer; therefore the number in the latter corresponding to $n - N$ may be obtained from the proportion

$$N' - N : n - N :: 100 : t.$$

If the thermometer be Fahrenheit's, $N' = 212$, $N = 32$;

$$\therefore\ t = \frac{100 . (n - 32)}{180} = \frac{5 . (n - 32)}{9},\ \text{as before.}$$

495. Cor. 2. If $x = 32$, $y = 212$, which is the case in Fahrenheit's,

$$t = 32 + \frac{180 . (n - N)}{N' - N}.$$

If it be the centesimal, $N = 0$, $N' = 100$;

$$\therefore\ t = 32 + \frac{9n}{5}.$$

496. It appears from experiment that the apparent expansion of mercury from the point of melting snow to that of boiling water is $\frac{1}{63}$ of its magnitude at the former of these temperatures; and the expansion for each degree is constant, that is, in the centesimal thermometer is $\frac{1}{6300}$ for every $\frac{1}{100}$th part of the interval.

This is the apparent expansion. For upon measuring the expansion of glass, it is found that the *real* expansion of mercury is $\frac{100}{5412}$ from melting snow to boiling water. This is greater than the apparent, as it ought to be; the latter being in fact the excess of the expansion of the mercury above that of glass.

497. It is important to remark, that the indications of the thermometer do not depend upon the *absolute* quantity of this expansion: for if this quantity were double or triple of that mentioned, provided it followed the same proportion in all temperatures, the number of degrees indicated by the thermometer would be the same under the same circumstances: the initial dimensions at the temperature of melting snow being the same, the expansions to the temperature of boiling water would in this case be doubled or tripled, and consequently the degrees also which are certain parts of that interval. Hence thermometers may be compared though they are constructed of different kinds of glass: for experiments shew that the expansions of mercury are exactly proportional to those of glass and other solids which melt only at high temperatures. The unequal expansion therefore of the different kinds of glass will cause a proportional change in the lengths of the intervals and of the intermediate degrees; but yet the same degree in each will correspond with the same temperature though the absolute lengths of the degree in different instruments be different.

498. Required the dimensions of the bulb of a thermometer, the length of its tube, and the quantity of mercury therein, supposing the graduation to extend from a^0 below the

temperature of melting snow, to b^0 above that of boiling water.

The apparent expansion of mercury only being considered, suppose the vessel containing it not to expand. Let the tube be divided into any number of equal portions, each of the capacity (m). 1st. To determine the capacity of the bulb. Let the bulb and a part of the tube be filled with mercury; and when plunged in melting snow let it stand at an altitude n above the bulb. Then if Nm represent the magnitude of the mercury in the bulb, $(N+n) \,.\, m$ will $=$ the whole magnitude of the mercury introduced.

Next observe the altitude A to which the mercury rises above the former in boiling water. Then Am is the apparent expansion of this quantity of mercury between the temperatures of melting snow and boiling water. Now this by experiment is $\frac{1}{\mu}$th part of the whole magnitude, and

$$\therefore \frac{(N+n) \,.\, m}{\mu} = Am;$$

$$\therefore N = \mu A - n;$$

whence Nm may be determined, the capacity of the bulb.

2. To find the column of mercury to be left in the tube.

Since the thermometer must shew a^0 below the temperature of melting snow; let $x =$ the length of the mercury to be left in the tube above the bulb, measured at that temperature.

$$\text{Then } mx = \text{its magnitude,}$$

$$\text{and } (N+x) \,.\, m = \text{the total magnitude;}$$

$$\therefore \frac{(N+x) \,.\, m}{\mu} = \text{its expansion between the fixed points,}$$

$$\text{and } \frac{N+x}{\mu} = \text{the length of the tube occupied;}$$

or if ϕ be the number of degrees between the fixed points of melting snow and boiling water,

$\frac{N+x}{\mu\phi}$ will be the length of 1°;

$\therefore \frac{a.(N+x)}{\mu\phi}$ = the length of a^0, which by supposition $= x$,

whence $x = \frac{aN}{\mu\phi - a}$ the length of the column to be left in the tube above the bulb.

3. To find the length of the tube.

As the scale is to extend to b^0 above boiling water, the number of degrees above the bulb is $a + b + \phi$; and the length of one degree is $\frac{N}{\mu\phi - a}$;

$\therefore$ the length of the tube $L = \frac{(a+b+\phi).N}{\mu\phi - a}$.

499. Cor. 1. Hence may be determined the number of degrees which a thermometer of given length will shew above the temperature of boiling water.

$$\text{For } b = \frac{L.(\mu\phi - a)}{N} - (\phi + a).$$

500. Cor. 2. If this expression be $= 0$, the thermometer will indicate no higher temperature than that of boiling water: if negative, it will indicate only b^0 below that point.

501. Cor. 3. $a = \frac{\mu\phi L - (\phi + b).N}{L + N}$.

If this expression be $= 0$, the mercury will descend in the thermometer only to the point of melting snow; and if negative, not so low by a^0.

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[*] The numbers a and b should always be taken a little larger than the number of degrees which are to be observed on the thermometer: as the cylinder will not be exact close to the bulb and to the upper extremity where it is hermetically sealed. The observations therefore should not be made near to those points.

502. Cor. 4. To determine the length of a column of mercury at the temperature of melting snow, to be left in the preceding cases, in the thermometer whose length is L, in order that it may shew also b^0 above the temperature of boiling water.

$$(498.)\quad x = \frac{aN}{\mu\phi - a},$$

$$\text{and}\ (501.)\quad \mu\phi - a = \frac{[(\mu+1).\phi+b].N}{L+N};$$

$$\text{whence}\ x = \frac{\mu\phi L - (\phi+b).N}{(\mu+1).\phi+b},\ \text{the length required.}$$

503. Cor. 5. If this value of x be negative, the mercury at the temperature of melting snow will enter the bulb, and leave there a vacuum of the magnitude mx.

504. To determine the capacity of the bulb, by weighing; without its being plunged into melting snow or boiling water.

Before the bulb is blown, let the tube be weighed when empty; and also when a column of mercury which occupies λ divisions has been introduced; the difference (w) of these weights is the weight of the mercury in the tube whose length is λ. If $\therefore$ w' = the weight of mercury in one division,

$$w = \lambda w'.$$

When the bulb is blown, let it be weighed again; and also when the bulb and a certain length n is filled with mercury. The difference (W) of these weights is the weight of the mercury introduced. Let Nw' = the weight of that in the bulb;

$$\therefore\ W = (N+n).w';$$

$$\text{whence}\ \frac{N+n}{\lambda} = \frac{W}{w};$$

$$\text{and}\ N = \frac{W\lambda}{w} - n;$$

whence Nm the capacity of the bulb may be found; and the other dimensions as before.

505. To determine the corrections for the observed points of melting snow and boiling water, when the bulb only is plunged in those temperatures.

Suppose the temperature of the atmosphere to be t^0 above that of melting snow; and $l=$ the length of the mercury in the tube not plunged; which may therefore be considered of the same temperature with the atmosphere. Let $x=$ the length it would have if plunged in the temperature of melting snow.

Then x by expanding would become $x.\left(1+\frac{t}{\mu\phi}\right)$,

which is $=l$, the length it is observed to have;

$$\therefore\ x=\frac{l}{1+\frac{t}{\mu\phi}}=l-\frac{lt}{\mu\phi+t};$$

the first term of which is the observed length; and the second the quantity to be subtracted from the observed length, in order to have the true point of melting snow on the tube.

Again, suppose the bulb and a part of the cylinder of mercury to be plunged in the temperature of boiling water; and $l'=$ the length of the column which remains in the air at t'^0 below that temperature. Let $x'=$ the length it would have if entirely immersed, and $x''=$ the length it would have at the temperature of melting snow;

$$\text{then } x''.\left(1+\frac{\phi}{\mu\phi}\right)=x',$$

$$\text{and } x''.\left(1+\frac{(\phi-t')}{\mu\phi}\right)=l',$$

since x'' expanding to $(\phi-t')^0$ becomes l';

whence

$$x'=\frac{l'.\left(1+\frac{\phi}{\mu\phi}\right)}{1+\frac{\phi-t'}{\mu\phi}}=\frac{(\mu+1).\phi l'}{(\mu+1).\phi-t'}=l'+\frac{l't'}{(\mu+1).\phi-t'};$$

the second term of which will express the quantity to be added to l', in order to have the real point which indicates the temperature of boiling water.*

506. Cor. If $t'=0$, the second term vanishes, and $x'=l'$, the mercury being at the temperature ϕ.

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* In all cases of accuracy these corrections are necessary; and most probably the little attention which has been paid to them has been the cause of the variations in the results of different observations.

SECT. IX.

507. IF any homogeneous body expand by the application of heat; its cubical expansion will be triple the linear expansion.

Let M be the magnitude of the body, and M' its magnitude when heated; and let l and l' be its lengths similarly measured in the two cases.

Since the body will preserve a similar form,

$$M : M' :: l^3 : l'^3;$$
$$\therefore\ M : M'-M :: l^3 : l'^3-l^3.$$

Hence
$$\frac{M'-M}{M}=\frac{l'^3-l^3}{l^3}=\frac{l'^2+ll'+l^2}{l^3}.(l'-l)$$
$$=\frac{3l.(l+x)+x^2}{l^3}.(l'-l) \text{ if } l'=l+x,$$
$$=3.\frac{l'-l}{l} \text{ nearly,}$$

since in bodies at temperatures distant from fusion, x is small compared with l.

But $\frac{M'-M}{M}$ will represent that part of the whole quantity by which the body has increased, or its cubical expansion; and $\frac{l'-l}{l}$ the linear expansion. Hence the cubical is triple the linear expansion.

508. COR. 1. Hence

$$M'=M.\left\{1+3.\frac{l'-l}{l}\right\}=M.\{1+3\lambda\},$$

if λ be assumed $=\frac{l'-l}{l}$.

509. Cor. 2. In solid bodies, whilst the temperature is between that of melting snow $(=z^0)$ and boiling water, the linear expansion $\frac{l'-l}{l}$ is *found* to be proportional to the number of degrees on the thermometer, reckoned from z. When the thermometer is at z, let $M=$ the magnitude of the body; and t^0 the altitude of the thermometer when the magnitude of the body (M') is required: then if the linear expansion for 1° be represented by e, that for t^0 will be $= et$.

$$\text{Hence } M'=M.\{1+3et\}=M.\{1+Et\}.$$

510. Cor. 3. If M' and M'' are the magnitudes of the body when the temperature is t' and t'',

$$M'=M.\{1+Et'\}, \text{ and } M''=M.\{1+Et''\},$$

supposing M the magnitude when the thermometer is at z;

$$\therefore\ M''=\frac{M'.\{1+Et''\}}{1+Et'}=M'.\left\{1+\frac{E.(t''-t')}{1+Et'}\right\};$$

and since the cubical expansion is small, Et' may be neglected in the approximation,

$$\text{whence } M''=M'.\{1+E.(t''-t')\},$$

that is, precisely the same result is obtained as if the expansion had been calculated from the temperature t' and magnitude M'; always with the same coefficient E.

511. Cor. 4. If the expansion be considerable, the approximations deduced above will not be sufficiently accurate. But since $\frac{M'-M}{M}=\frac{l'^2+ll'+l^2}{l^3}.(l'-l)$ if the expansion be supposed uniform; let $\frac{l'-l}{l}=et$, where e represents the linear expansion for the increase of 1° of temperature; then $l'=l.\{1+et\}$;

$$\therefore\ \frac{M'-M}{M}=\{3+3et+e^2t^2\}.et$$

$$=Et.\left\{1+\frac{Et}{3}+\frac{E^2t^2}{27}\right\} \text{ if } E=3e.$$

$$\text{Hence } M' = M.\left\{1 + Et + \frac{E^2t^2}{3} + \frac{E^3t^3}{27}\right\},$$

the two first terms of which are the same with those deduced above. In solid bodies there are no cases in which the two others are *sensible.* In considering the expansion of fluids, they may perhaps be necessary, E being greater than in the case of solids.

512. If an elastic fluid be inclosed in a regular graduated vessel, and confined by a piston placed horizontally, or a small quantity of mercury; and the temperature of the fluid and vessel be changed; to find its cubical expansion, corrected for the expansion of the vessel.

Let M be the number of divisions occupied, which will therefore measure its magnitude; and suppose the temperature to become t^0; and let x represent the cubical expansion of the fluid from z to t^0,

its magnitude becomes $M.\{1+x\}$.

Let M' represent the number of divisions it now occupies in the vessel. But as the vessel itself expands, each of these divisions has a capacity different from what it had at the initial temperature. Let therefore E represent the cubical expansion of the substance of which the vessel is composed, for 1° of the thermometer; then M' divisions at the temperature t will be equivalent to $M'.\{1+Et\}$ primitive divisions; this therefore will represent the new magnitude of the fluid, expressed in parts of the primitive divisions; that is,

$$M.\{1+x\} = M'.\{1+Et\};$$

$$\text{whence } x = \frac{M'-M}{M} + \frac{M'Et}{M};$$

where the first term represents the cubical expansion of the fluid supposing the vessel not to expand; and the second the correction to be applied for its expansion.

513. Cor. 1. The proof will be similar for any fluid.

514. Cor. 2. The proposition has been proved on the supposition that the pressure of the atmosphere remained the same. Supposing that not to be the case, let p represent the pressure when the fluid occupied M divisions; if this be reduced to its magnitude under a constant pressure a (451) M will become $\frac{Mp}{a}$; and supposing p' the pressure of the atmosphere, when the fluid is at the temperature t^0, M' will become in the same manner $\frac{M'p'}{a}$. Let these values therefore be substituted in the preceding equation, and

$$x = \frac{M'p' - Mp}{Mp} + \frac{M'p'Et}{Mp}.$$

515. Cor. 3. When x is determined for t^0, it may be continued for $2t$, $3t$, &c.: and by a comparison of the values it will be seen whether the expansion is uniform or not. If it be uniform, the expansion will be $2x$, $3x$, &c. proportional to the difference of temperature.

516. By this mode the following conclusions have been deduced:

1. All the permanent gases exposed to equal temperatures, under the same pressure, expand exactly by the same quantity.

2. The extent of their common expansion from the temperature of melting snow to that of boiling water is $= 0.375$ of their primitive magnitude; the pressure being constant.

3. Between these two limits the expansion of gas is exactly proportional to the expansion of mercury. Whence it follows that for every degree of the thermometer, whose scale is n^0, and under the same pressure, all the gasses expand by a quantity $= \frac{1}{n}(0.375)$ of the magnitude which they had at the temperature of melting snow.

517. If a fluid be weighed at initial temperature, and again at any other temperature; its expansion may be determined.

Let M be the content of the vessel in which the fluid is weighed, at initial temperature; and $W=$ the weight of the fluid which it holds when full, at that temperature.

When the temperature is changed to t^0, the quantity of the fluid which the vessel will contain, is changed; let therefore $W'=$its weight; then the space which this would occupy at initial temperature $=\frac{MW'}{W}$; which is what now occupies the whole vessel.

Let x represent its expansion between the two temperatures; this should have changed into $\frac{MW'}{W}.(1+x)$.

Also the capacity of the vessel becomes $M.\{1+Et\}$;

and as the fluid fills the vessel, $\frac{MW'}{W}.(1+x)=M.\{1+Et\}$;

$$\therefore\ x=\frac{W-W'}{W'}+\frac{WEt}{W'};$$

which is similar to the former expression.

518. To determine the expansion of a fluid, from weighing a given solid body in it at different temperatures.

A body immersed in a fluid loses a weight equal to the weight of the fluid displaced (110); and this will be greater or less according to the temperature.

Suppose then $M=$the magnitude of the body at initial temperature, and $W=$ the loss of weight which it would sustain at that temperature, and therefore equal to the weight of the magnitude M of the fluid: and let $W'=$ the loss of weight sustained by the body when the temperature is t, and therefore equal to the weight of a quantity of the fluid of the same magnitude with the body.

But the magnitude of the body becomes $M . \{1+Et\}$, which therefore is the magnitude of the portion of fluid whose weight is W'. Hence the original mass whose weight was W occupies a space $=\frac{MW}{W'} . \{1+Et\}$. And if x be the cubical expansion of the fluid between the two temperatures, M has changed to $M . (1+x)$;

$$\therefore\ M . (1+x) = \frac{MW}{W'} . \{1+Et\};$$

$$\text{and}\ x = \frac{W-W'}{W'} + \frac{WEt}{W'},$$

which is similar to the preceding.

519. In applying these different methods to ascertain the expansion of water, we arrive at a remarkable result, that in cooling it does not contract in an equable manner. Its contraction diminishes for every degree, as the thermometer descends to about 40°. But beyond that limit, if the thermometer descends, the magnitude remains some time invariable, after which it expands instead of contracting.

520. Having observed the apparent weight of distilled water contained in an expansive vessel, under known circumstances; to determine the capacity of the vessel, at the temperature z.

Let M' be the number of cubical inches at max. condensation, which when the thermometer rises to t^0, would fill the vessel. Its content $= M' . (1+x)$, if x represent the expansion of water from max. condensation to t^0.

Let X be its magnitude at the temperature z,

at the temperature t^0 it becomes $X . \{1+Et\}$.

But the vessel is now filled with $M' . (1+x)$;

$$\therefore\ X . \{1+Et\} = M' . (1+x),$$

$$\text{or}\ X = \frac{M' . (1+x)}{1+Et}.$$

If then M' was known, X would be determined. To determine M', let the vessel be weighed full of air and open; and again full of water; and let M be the difference of the observed weights; it will be the difference between the weight of water it contains at the temperature t, and an equal quantity of air. Let the specific gravities of the water and air be as $1 : a$,

$$\text{then } M' . (1-a) = M,$$

$$\text{and } \therefore M' = \frac{M}{1-a};$$

$$\text{whence } X = \frac{M . (1+x)}{(1-a) . (1+Et)}.$$

521. Cor. The second member of the equation, containing only quantities which are given from observation, may be reduced to a form more easy of application, by remarking that a, x, and Et are small fractions in those circumstances under which experiments are usually made; and therefore X will not differ much from M, which may therefore be separated from the small corrections, under the form

$$x = M + Mx + \frac{M . (1+x) . (a - Et + aEt)}{(1-a) . (1+Et)}.$$

522. If ϵ_t represent the real expansion of an unit of magnitude reckoned from z to t^0 of the mercurial thermometer, and e_t the apparent expansion; and E the cubical expansion of the substance of the vessel in which the fluid is observed,

$$e_t = \epsilon_t - Et.$$

Suppose when $t = 0$, the fluid occupies M divisions of the vessel; and at t^0 the number to be X; this will be the apparent magnitude. But (511) the real magnitude will be

$$= X . \left\{1 + Et + \frac{E^2 t^2}{3}\right\}.$$

And this by the supposition is $= M . \{1 + \epsilon_t\}$.

$$\text{Hence } X = \frac{M . \{1 + \epsilon_t\}}{1 + Et + \frac{1}{3}E^2t^2} = M + \frac{M . \{\epsilon_t - Et - \frac{1}{3}E^2t^2\}}{1 + Et + \frac{1}{3}E^2t^2};$$

$$\text{and } X - M = \frac{M.\{\epsilon_t - Et - \frac{1}{3}E^2t^2\}}{1 + Et + \frac{1}{3}E^2t^2},$$

$$\text{that is, } Me_t = \frac{M.\{\epsilon_t - Et - \frac{1}{3}E^2t^2\}}{1 + Et + \frac{1}{3}E^2t^2},$$

$$\text{or } e_t = \frac{\epsilon_t - Et - \frac{1}{3}E^2t^2}{1 + Et + \frac{1}{3}E^2t^2}.$$

Now in the most exact observations on the expansions of fluids in glass between the temperatures of 5^0 and 212^0 of Fahrenheit, the terms affected with $\frac{1}{3}E^2$ will be insensible;

$$\therefore\ e_t = \frac{\epsilon_t - Et}{1 + Et} = \epsilon_t - Et,$$

neglecting E^2 and $E\epsilon_t$.

523. Hence may be measured the difference of the expansion of solids.

Observe the apparent expansion of a fluid in vessels made of the substances which it is wished to try; and let the observations be made between constant temperatures; and let E, E' be the cubical expansions.

$$\text{Since } e_t = \frac{\epsilon_t - Et}{1 + Et};$$

$$\therefore\ e_t.(1 + Et) = \epsilon_t - Et;$$

$$\text{and } (1 + e_t).Et = \epsilon_t - e_t,$$

the difference between the real and apparent expansions.

Let now a vessel of a different substance, but containing the same fluid, be exposed to the same temperatures,

$$\text{then } (1 + e'_t).E't = \epsilon_t - e'_t;$$

$$\text{whence } (1 + e'_t).E't - (1 + e_t).Et = e_t - e'_t;$$

$$\text{and } \therefore\ E' = E + \frac{(e_t - e'_t).(1 + Et)}{t.(1 + e'_t)}.$$

Now the expansions of metals are sufficiently known, to allow their being used to calculate the small correction dependent upon E in the second member of the equation. Substituting therefore for e_t, e'_t, and t, their observed values, $E' - E$ will be known.

524. Cor. 2. To determine the expansion of solid bodies; that of one fluid between two fixed temperatures being given.

Expose a vessel filled with the fluid to those two temperatures; and let $t^0 =$ their difference, the apparent expansion e_t may be observed; and the real one ϵ_t is given;

$$\text{hence } E = \frac{\epsilon_t - e_t}{t.(1 + e_t)}.$$

If ϵ_t is not known, it may be determined from the apparent expansion of the fluid observed between the same limits in a vessel of glass or metal exactly graduated. For the absolute expansions of glass and the greater part of metals have been observed, and determined with so much care that they may be considered as *exactly* known.

525. To find the specific gravity of a fluid by means of the hydrometer.

Let W be the absolute weight of the instrument *in vacuo*, that is, corrected for the weight of the atmosphere which it displaces (115). Suppose it plunged in water at the max. condensation, and to sink to a *given* point, marked on the stem. Then since the weight of a floating body is equal to the weight of a quantity of fluid of the same magnitude with the part immersed (101), the weight of a quantity of water equal in magnitude to the part immersed will be W. Let its magnitude be M, which will be known, and therefore the magnitude of the portion of the instrument immersed. Suppose M' to represent the magnitude of the same portion at the temperature z; and τ the temperature of max. condensation; then (509),

$$\text{at that temperature its magnitude is} = M'.(1 + E\tau);$$
$$\therefore\ M = M'.(1 + E\tau).$$

Determining thus for the temperature z, the magnitude of the portion constantly immersed, we find the magnitude of the same portion for any other temperature t^0, $= M'.(1 + Et)$.

Suppose it now plunged in a fluid whose specific gravity s at the temperature z is required, that is, in one, a cubic foot of which weighs s ounces at the temperature z, and whose expansion from that point to $t^0 = \epsilon$; a cubic foot of the fluid taken at the latter temperature will weigh $\frac{s}{1+\epsilon}$ ounces. And as the instrument displaces $M'.(1+Et)$ of these feet, the whole weight of the fluid displaced will be $\frac{M's.(1+Et)}{1+\epsilon}$. Now this weight is given by observation $= W + W'$; W being the primitive and absolute weight of the instrument, and W' the weight to be added to make it sink to the *given* point;

$$\text{whence } \frac{M's.(1+Et)}{1+\epsilon} = W + W',$$

$$\text{and } s = \frac{(W+W').(1+\epsilon)}{M'.(1+Et)}.$$

526. To determine the specific gravities of fluids.

Let W be the apparent weight of the fluid which fills the vessel at the temperature t^0; a the weight of the air previously contained in the vessel at the same temperature; $\therefore W + a =$ the absolute weight of the fluid *in vacuo* (115).

In order to determine the absolute weight of that portion of the fluid which would fill the vessel at the temperature z; let W' represent it; then the weight of this fluid at any temperature t is equal to $\frac{W'.(1+Et)}{1+\epsilon}$, if ϵ represent the expansion of the fluid from z to t^0.

But this is $= W + a$;

$$\therefore \frac{W'.(1+Et)}{1+\epsilon} = W + a,$$

$$\text{and } W' = \frac{(W+a).(1+\epsilon)}{1+Et}.$$

Divide this by M the capacity of the vessel at the temperature z, and the specific gravity will be

$$\frac{(W+a).(1+\epsilon)}{M.(1+Et)}.$$

527. If the specific gravity (s) of a fluid for the temperature z be known, its specific gravity for any other temperature may be found.

Let $\epsilon =$ the expansion of the fluid reckoned from z to t^0, and $\epsilon' =$ the expansion of water from max. condensation to t^0.

Then the weight of a cubic foot of fluid $= \dfrac{s}{1+\epsilon}$,

and of water $= \dfrac{1}{1+\epsilon'}$;

∴ the measure of the ratio of the first to the second is

$$\frac{s.(1+\epsilon')}{1+\epsilon}, \quad (8).$$

528. Cor. In the case of mercury, the expansion is uniform and $= \dfrac{t}{9742}$ for every temperature t; therefore the measure of the ratio of the weights will be $\dfrac{s.(1+\epsilon')}{1+\dfrac{t}{9742}}$. But at max. condensation $\epsilon' = 0$, and let λ be the value of t;

∴ the specific gravity of mercury $= \dfrac{s}{1+\dfrac{\lambda}{9742}} = s - \dfrac{\lambda s}{9742}$.

529. To determine the specific gravity of a solid body.

Let the solid be weighed in air, and in a known fluid at the temperature t; and let $M =$ the magnitude of that solid in cubic feet; $s =$ the absolute weight of one cubic foot at the temperature z; E its expansion for one degree of the thermometer. The weight of a cubical foot of this body at the temperature t, is $\frac{s}{1+Et}$, the expansion of solids between the temperatures of melting snow and boiling water being considered uniform. The whole weight therefore will be $\frac{Ms}{1+Et}$.

In the same way σ being the weight of a cubical foot of water at the temperature z, and ϵ' the expansion of water from z to t^0; the weight of the magnitude M at the temperature t^0 will be $\frac{M\sigma}{1+\epsilon'}$, and the weight of an equal magnitude of air will be $\frac{M\sigma a}{1+\epsilon'}$, if a represent the measure of the ratio of the weight of air to that of water under the circumstances of the experiment. And $\frac{M\sigma'}{1+\epsilon}$ will be the weight of the same magnitude of any fluid, the expansion of which from z to t^0 is ϵ, and a cubic foot of which would weigh σ' ounces at the temperature z.

Now these different weights are what the solid body has lost when it has been weighed successively in air and in the fluid (110); so that in subtracting them from the absolute weight $\frac{Ms}{1+Et}$, we obtain the apparent weights, which call S and S'.

$$\text{Hence } \frac{Ms}{1+Et} - \frac{M\sigma a}{1+\epsilon'} = S,$$

$$\text{and } \frac{Ms}{1+Et} - \frac{M\sigma'}{1+\epsilon} = S';$$

$$\text{and dividing the two equations } \frac{\frac{s}{1+Et} - \frac{\sigma'}{1+\epsilon}}{\frac{s}{1+Et} - \frac{\sigma a}{1+\epsilon'}} = \frac{S'}{S};$$

$$\text{and } \frac{s}{1+Et} = \frac{\frac{S\sigma'}{1+\epsilon} - \frac{S'\sigma a}{1+\epsilon'}}{S-S'},$$

which is the weight of a cubical foot of the body at the temperature t.

$$\text{and } s = \frac{(1+Et).\left[\frac{S\sigma'}{1+\epsilon} - \frac{S'\sigma a}{1+\epsilon'}\right]}{S-S'},$$

the weight of a cubical foot of the body at the temperature z.

530. Cor. 1. If the body is weighed successively in water and air of the same temperature; $\sigma'=\sigma$, $\epsilon=\epsilon'$, and the expression becomes $s = \dfrac{(1+Et).\sigma.(S-S'a)}{(S-S').(1+\epsilon')}$.

531. Cor. 2. If we neglect these corrections, as well those which arise from the expansion of the fluid and the solid, as the weight of the air displaced; E, ϵ', a, become $=0$;

$$\text{whence } s = \frac{\sigma S}{S-S'}, \text{ or } \frac{s}{\sigma} = \frac{S}{S-S'};$$

the first of which expressions gives the weight of a cubic foot of the fluid; the second its specific gravity compared with that of water. This is the common approximation.

532. Cor. 3. In general, when a very accurate value of s is obtained; divide it by σ the weight of a cubical foot of water at the temperature z, and the quotient $\dfrac{s}{\sigma}$ will express the ratio of the weights of equal magnitudes of the body and water at that temperature. If we wish to refer their specific gravity to the temperature τ of max. condensation of water, it will only be necessary to reduce the weight s to that temperature by dividing it by $1+E\tau$; then $\dfrac{s}{1+E\tau}$ will be the specific gravity of the fluid.

533. Cor. 4. If we make use of a jar full of a known fluid, in which a solid body is introduced after being weighed in air, the formula becomes a little different, according to the manner in which the operation is performed. Suppose therefore we weigh

1. The solid body,
2. The jar filled with the fluid,
3. The jar filled with the solid and fluid.

Retaining the same notation, and calling M' the capacity of the vessel at the temperature at which the operation is performed, W' the apparent weight of the fluid in the second operation, and Σ the sum of the weights of the fluid and solid in the third,

$$1. \quad \frac{Ms}{1+Et} - \frac{M\sigma a}{1+\epsilon'} = S,$$

$$2. \quad \frac{M'\sigma'}{1+\epsilon} - \frac{M'\sigma a}{1+\epsilon'} = W',$$

$$3. \quad \frac{(M'-M).\sigma'}{1+\epsilon} + \frac{Ms}{1+Et} - \frac{M'\sigma a}{1+\epsilon'} = \Sigma.$$

Subtracting the second equation from the third,

$$\frac{Ms}{1+Et} - \frac{M\sigma'}{1+\epsilon} = \Sigma - W'.$$

$$\text{But} \quad \frac{Ms}{1+Et} - \frac{M\sigma a}{1+\epsilon'} = S,$$

which are similar to those in the proposition (529), with $\Sigma - W'$ in the place of S'. Hence

$$s = \frac{(1+Et).\left[\frac{S\sigma'}{1+\epsilon} - \frac{(\Sigma - W').\sigma a}{1+\epsilon'}\right]}{S - \Sigma + W'},$$

534. Cor. 5. If the fluid in which the body is immersed be pure water, $\sigma' = \sigma$, $\epsilon = \epsilon'$,

$$\text{and } s = \frac{\sigma . (1 + Et) . [S - (\Sigma - W')a]}{(1 + \epsilon') . (S - \Sigma + W')}:$$

$$\text{and neglecting the corrections } \frac{s}{\sigma} = \frac{S}{S - \Sigma + W'}.$$

535. Cor. 6. Suppose the body to be weighed in air: then fill the vessel with distilled water at a known temperature; place the body with the vessel thus filled in one of the scales of a balance, and weigh the whole. Then immerse the solid in the vessel, whereby a quantity of water will be expelled, and let the vessel be properly closed so that no air be left within. As it is now lighter by the quantity of water expelled, determine that weight by restoring the equilibrium. Hence three equations will be determined for ascertaining the specific gravity. Let Σ' = the weight of the system in the second case; then

1. When the body is weighed in air,

$$\frac{Ms}{1 + Et} - \frac{M\sigma a}{1 + \epsilon'} = S.$$

2. When the body is weighed with the vessel full of fluid,

$$\frac{Ms}{1 + Et} - \frac{M\sigma a}{1 + \epsilon'} + \frac{M'\sigma'}{1 + \epsilon} - \frac{M'\sigma a}{1 + \epsilon'} = \Sigma'.$$

3. When the vessel is filled with the fluid and solid,

$$\frac{(M' - M) . \sigma}{1 + \epsilon} + \frac{Ms}{1 + Et} - \frac{M'\sigma a}{1 + \epsilon'} = \Sigma.$$

Hence $\frac{M'\sigma'}{1 + \epsilon} - \frac{M'\sigma a}{1 + \epsilon'} = \Sigma' - S$, and subtracting this from the third

$$\frac{Ms}{1 + Et} - \frac{M\sigma'}{1 + \epsilon} = \Sigma - \Sigma' + S,$$

which is the same with the first (533) as before, with $\Sigma' - S$ in the place of W'; and these will correspond with the equations of the proposition (529) by changing S' into $\Sigma - \Sigma' + S$.

536. Cor. 7. If the corrections be neglected, the specific gravity which was $\frac{S}{S - \Sigma + W'}$ will have become

$$\frac{S}{S - \Sigma + \Sigma' - S} \text{ or } \frac{S}{\Sigma' - \Sigma}.$$

SECT. X.

537. DEF. THE *Common Barometer* is a cylindrical glass tube, about 31 inches long, exhausted of air and filled with mercury, hermetically sealed at one end, and immersed vertically at the other into a basin of mercury when the mercury in the tube is found to subside to a certain height, leaving a vacuum at the upper end.

538. DEF. The altitude, at which the mercury is sustained in the tube, above the surface of that in the basin, is called the *standard altitude*.

539. DEF. This altitude is always between 28 and 31 inches; the difference between these altitudes is called the *scale of variation*.

540. The mercury is supported in the tube by the action of the air.

For let the barometer be placed in the receiver of an air-pump, and the air be exhausted; the mercury will gradually descend to a level with that in the basin: and if the air be re-admitted, it will rise again in the tube.

541. This effect is produced by the air's *pressure*.

For it is evident that the pressure upon every portion of the surface of the mercury in the vessel is the weight of the incumbent column of atmosphere; and the fluid being at rest, action and re-action are equal and opposite; therefore the pressure *upwards* against every portion is such as will support the weight of the corresponding column of atmosphere. And consequently the pressure upwards against that portion which is immediately below the surface of the barometer is such as would support a column of atmosphere whose base is equal to the orifice of the tube: which pressure must be counteracted by the weight of the mercury in the barometer; since in this case only will the pressures upon equal portions of the surface of the mercury be equal.

542. COR. 1. Since the weight of the column of mercury *cæt. par.* is as its length; the weight of the air is proportional to the altitude of the mercury in the barometer.

543. COR. 2. The standard altitude will be the same at the same place in all tubes, whatever be the magnitude or figure of the section perpendicular to the axis, or whatever their position, provided the tube be not so slender as to expose the mercury to a sensible alteration from the capillary attraction; and the mercury be supposed not to expand.

544. COR. 3. Since mercury expands by heat, the dimensions of a given quantity of it will be variable unless the temperature of the air remain the same; and its altitude in the barometer will cease to be a correct measure of the weight of the atmosphere. These altitudes therefore must be reduced to what they would be under the same temperature.

Let $l=$ the altitude of a column observed at the temperature t. Suppose the temperature to be reduced to that of melting snow, the weight of the atmosphere remaining the same; then the column will be contracted; suppose the height to become l'[*]. Then l and l' represent the magnitudes of two cylinders of mercury of equal weights and bases;

$$\therefore\ l=l'.\left(1+\frac{t}{a}\right)\dagger;$$

$$\therefore\ l'=\frac{l}{1+\frac{t}{a}}=l-\frac{lt}{a+t},$$

---

[*] The tube will also be contracted; but this will have no influence on the *height* of the mercury sustained by the air's pressure.

[†] The value of a in Fahrenheit's thermometer is 9742; in the centigrade 5412.

the second term expressing the correction to be applied to the height l in order to obtain that which would be observed under the same pressure at the temperature of melting snow.

545. Cor. 4. In the temperatures in which observations are commonly made, t is small compared with a, and the second term of the denominator may be neglected; in which case

$$l' = l - \frac{lt}{a}.$$

546. Cor. 5. If the diameters of the tubes be different, the heights of the mercurial columns in barometers equally well constructed, will be different at the same place. This however is not very sensible beyond a certain diameter, and may be counterbalanced by having the barometer in the form of an inverted syphon.

547. Cor. 6. The pressure on each point of the earth's surface being equal to the weight of a column of mercury, the mean height of which is known; the weight of the mass of air which surrounds the earth may be determined.

Let $R =$ the radius of the earth, and $r =$ the height of the mercury, $s =$ its specific gravity;

$$\begin{aligned} \text{the weight required} &= s \, . \left[\frac{4\pi . (R+r)^3}{3} - \frac{4\pi R^3}{3}\right] \\ &= \frac{4s\pi}{3} . (3R^2r + 3Rr^2 + r^3) \\ &= 4s\pi . (R^2r + Rr^2 + \tfrac{1}{3}r^3) \\ &= 4s\pi R^2 r, \text{ nearly,} \end{aligned}$$

which Cotes calculated to be equal to the weight of a globe of lead 60 miles in diameter.

548. Cor. 7. The altitude of the mercury in the barometer at any height above the earth's surface is less than the standard altitude at the earth's surface.

The suspension of mercury in the barometer being caused by the pressure or weight of the atmosphere, if part of that weight be removed, the altitude of the mercury will be diminished. Now when the barometer is above the earth's surface, the air beneath it has no effect in supporting the mercury; the pressure therefore being diminished, the altitude of the mercury is diminished by the length of a column whose weight is equal to that of the air between the barometer and the surface of the earth.

549. Cor. 8. Hence also, the altitude of the mercury in the barometer will be a measure of the air's pressure and elastic force at different heights above the earth's surface.

550. If a barometer be suspended from the extremity of a balance; the weight necessary to counterpoise it, exclusive of the tube, is equal to the weight of the mercury contained in the barometer.

For when the tube is empty, the air presses equally on the top internally and externally. But when filled with mercury, the pressure on the top externally remains, whilst a vacuum is within. Hence the pressure of the atmosphere on the top of the tube externally will be the same as on the corresponding horizontal section at the surface of the mercury in the reservoir, that is, it will be equal to the weight of the mercury in the tube.

551. The common construction is liable to an imperfection, which prevents the altitudes of the columns of mercury indicated by the scale, from being exactly proportional to the different pressures of the air; for as that column ascends or descends, it causes a small portion of the mercury contained in the basin, to pass into or out of the tube, and changes the place of its upper surface from that which it would occupy were it constantly to correspond to the zero of the scale. This imperfection is the less perceptible, as the basin has more breadth about the place of the line of level.

552. Different methods have been contrived to remedy this imperfection. In some barometers the scale is rendered

moveable in the direction of its height; in such a manner, that with the aid of a micrometer-screw, the line of level can always be brought exactly opposite the zero of the scale. In such a case there is substituted for the basin a portion of the tube, which is turned up again at its lower part, whence the sensible variation of the level which results may always be corrected by the motion of the scale.

In other cases, the basin of the barometer is entirely immersed in a cistern of mercury. When an observation is to be made, the barometer with its basin is elevated above the surrounding mercury; and the basin being always found full, the line of level given by the upper surface of the mercury retains a fixed position with respect to the graduation.

553. When the mercury in the tube of a barometer sinks, and the surface of that in the basin rises; to determine the correction.

Let $a =$ the section of the tube, and $b =$ that of the basin, supposed cylindrical. Let the mercury descend through a space x in the tube, and ascend through a space y in the basin;

$$\therefore\ ax = y\,.\,(b-a),$$

$$\text{and } y = \frac{ax}{b-a}.$$

And the real diminution of the height of the mercury, or the whole difference of the altitudes

$$= x + y = x + \frac{ax}{b-a} = \frac{bx}{b-a} = z \text{ (suppose)};$$

therefore the apparent diminution of height : the real diminution :: $b-a : b$.

554. Cor. Hence $x = \frac{b-a}{b}\,.\,z$. Suppose $z =$ one inch, and the area of the basin ten times that of the tube;

$$\therefore\ x = \frac{b-a}{b} \text{ inches} = \tfrac{9}{10} \text{ inch}.$$

Hence the barometrical inch must be $\frac{9}{10}$ths of a real inch, in order that the graduation on the scale may show the real depression.

555. The variations in an inclined barometer are greater than the corresponding variations in an upright one.

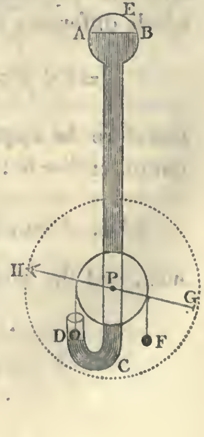
Since the pressure of fluids is proportional to their depths, the perpendicular altitudes of the columns in the two barometers will be equal. Suppose L and H to be the length and altitude of the column in the inclined barometer; and in consequence of some change in the atmosphere, to become l and h; then since the inclination remains the same

$$L : H :: l : h;$$

$$\text{whence } L-l : H-h :: L : H :: 1 : \sin\theta,$$

if $\theta =$ the inclination; that is, the variations are in the ratio of $1 : \sin\theta$; and therefore greater in the inclined than in the upright barometer.

556. The advantage of this barometer is that the scale of variation may be made of any magnitude: the disadvantage, that owing to the increased friction of the mercury upon the glass, the height does not vary with any slight change of the air. The column of mercury moreover is apt to break in the tube, or part of it to be left behind upon any considerable descent.



557. The *Wheel Barometer* is a compound tube $ABCD$ open at D, and closed at the upper part; the diameter of the upper part AEB being much greater than that of the tube. It is filled with mercury from D to AB; the portion AEB being a vacuum. Upon the surface of the mercury in the bent leg is an iron ball D connected with another F rather lighter, by a string passing over a pulley P. As the ball at D rises and falls with the mercury, the string turns the pulley, and an

index GH fixed to it, which points to the different parts of a graduated circle.

It is evident, that by increasing the diameter of the circle, this contrivance will shew the minutest variations of the air. But unless it be made with the greatest accuracy, the friction will render it useless.

The string is also affected by the variation of the humidity of the air.

558. The heights of mountains may be determined by means of the barometer and thermometer.

It is found by experiment that under the same pressure air expands uniformly between the temperatures of melting snow and boiling water. Let $\epsilon =$ the expansion for each degree of the thermometer, $e =$ the ratio of the elasticity of air to its density at the temperature of melting snow; then the bulk at the temperature x will be increased, and therefore the density diminished in the ratio of $1+\epsilon x : 1$;
$\therefore e \cdot (1+\epsilon x) =$ the ratio of the elasticity to the density at the temperature x.

Now let $p =$ the elasticity, $g =$ the force of gravity, at the earth's surface,
$P =$ the elasticity, $D =$ the density at temperature x, $G =$ the force of gravity, at the altitude z above the earth;

$$\text{then } \frac{P}{D} = e \cdot (1+\epsilon x).$$

Since $P - dP =$ the pressure at the altitude $(z+dz)$,
and $P =$ that at the altitude z;

$\therefore -dP =$ the difference of pressures $= DG\,dz = \dfrac{Dgr^2dz}{(r+z)^2}$,

if $r =$ the radius of the earth;

$$\therefore -\frac{dP}{P} = \frac{gr^2}{e \cdot (1+\epsilon x)} \times \frac{dz}{(r+z)^2},$$

$$\text{and} \quad \log. P = \frac{gr^2}{\mu e.(1+\epsilon x)} \times \frac{1}{r+z} + C,$$

μ being $= 2.30259$;

$$\text{and } \log. p = \frac{gr^2}{\mu e.(1+\epsilon x)} . \frac{1}{r} + C;$$

$$\therefore \log. \frac{p}{P} = \frac{gr^2}{\mu e.(1+\epsilon x)} . \left(\frac{1}{r} - \frac{1}{r+z}\right)$$

$$= \frac{gr}{\mu e.(1+\epsilon x)} . \frac{z}{r+z}.$$

It appears also from experiment that mercury contracts uniformly as its temperature decreases.

Let $H =$ the height of the barometer,
$M =$ the density of mercury,
and $T =$ the temperature of mercury,
} at the earth's surface,

and H', M', T' the same quantities respectively at the altitude z; and $\frac{1}{\beta} =$ the condensation of mercury for one degree of the thermometer;

$$\therefore M' = M . \left(1 + \frac{T - T'}{\beta}\right),$$

$$\text{and } P = M'GH' = \frac{gr^2}{(r+z)^2} . M . \left(1 + \frac{T-T'}{\beta}\right) . H',$$

$$\text{and } p = gMH;$$

$$\therefore \log. \frac{p}{P} = \log. \left(\frac{r+z}{r}\right)^2 . \frac{H}{\left(1 + \frac{T-T'}{\beta}\right) . H'}$$

$$= \log. \frac{H}{\left(1 + \frac{T-T'}{\beta}\right) . H'} + 2 \log. \left(1 + \frac{z}{r}\right).$$

Equating this with the preceding value of $\log. \frac{p}{P}$;

$$\therefore\ \frac{gr}{\mu e.(1+\epsilon x)}.\frac{z}{r+z}=\log.\frac{H}{\left(1+\frac{T-T'}{\beta}\right).H'}+2\log.\left(1+\frac{z}{r}\right),$$

$$\therefore\ z=\frac{\mu e.(1+\epsilon x)}{g}.\left\{\log.\frac{H}{\left(1+\frac{T-T'}{\beta}\right).H'}+2\log.\left(1+\frac{z}{r}\right)\right\}.\left(1+\frac{z}{r}\right),$$

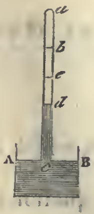
in which expression the quantity sought is on both sides of the equation; but z being small compared with r, $\frac{z}{r}$ may be neglected, and thus a first approximate value of z obtained, which being substituted for z on the right hand side of the equation, a nearer value of z will be obtained.

559. Cor. The temperature x has here been supposed to remain the same throughout the whole column z, whereas in fact it varies, and the law of its variation is subject to many irregularities. But for small heights, if altitudes be taken in arithmetic progression, it is a very slowly decreasing arithmetic progression. If therefore t and t' be the temperatures at the earth's surface and at the altitude z above it, the result will be nearly accurate, if we assume $x=\frac{1}{2}.(t+t')$. This supposition increases the temperature of the upper strata whilst it diminishes that in the lower, and thus produces a kind of compensation which for small heights is not far from the truth. The formula thus becomes

$$z=\frac{\mu e}{g}.\left\{1+\tfrac{1}{2}\epsilon(t+t')\right\}.\left\{\log.\frac{H}{H'.\left(1+\frac{T-T'}{\beta}\right)}\right.$$
$$\left.+2\log.\left(1+\frac{z}{r}\right)\right\}.\left(1+\frac{z}{r}\right).$$

560. If the tube of a barometer be perfectly cylindrical, and in part only filled with mercury, and then its open end be immersed in a basin of the same fluid, the mercury will sink below the standard altitude: and the standard altitude

will be to the depression below that altitude as the space occupied by the air after immersion is to the space occupied before.



Let ab be equal to the space occupied by the air before the tube is immersed, or when the air is in its natural state, ac being the whole length of the tube. On immersion let the mercury sink to d, and the air occupy the space ad. Since (457) the elasticity varies inversely as the space occupied by the same quantity,

the elasticity (E) of the air in ab : that in ad (E') :: $ad : ab$.

Let ce be the standard altitude. Then the elasticity of the air in ab (538) is such as would support a column of mercury ce; and the elasticity in ad is such as would support a column of mercury de; for it depresses the mercury to d;

$$\therefore E : E' :: ec : ed;$$

$$\text{whence } ec : ed :: ad : ab.$$

561. Cor. 1. Let the standard altitude $= h$, $ac = l$, $cd = x$, $ab = y$;

$$\text{then } h : h - x :: l - x : y,$$

$$\text{or } x^2 - (h + l).x = h.(y - l),$$

$$\text{and } x = \tfrac{1}{2}.\left(h + l \pm \sqrt{4h.(y - l) + (h + l)^2}\right)$$

in which y cannot be greater than l; its limits are o and l.

Let $y = o$; $\therefore x = h$ or l. The former is the case of a perfect barometer: the latter inadmissible, unless l be *not greater* than h; since the mercury cannot rise higher than in a perfect barometer.

Let $y = l$; $\therefore x = h + l$, or o; the former of which is impossible, if l is not $= o$, in which case there is no barometer. In the second case the mercury does not rise in the tube.

If y be assumed between the values of o and l, the quantity under the radical being real, if the upper sign be used, x will be greater than h, which is impossible. The value therefore of x is

$$\tfrac{1}{2}(h + l) - \tfrac{1}{2}\sqrt{(h + l)^2 - 4h.(l - y)}.$$

562. In the application of Algebra to physical problems, it frequently happens that the algebraic enunciation is common to the problem proposed, and to others which are not in view. In the present instance, we have the solution of another problem, in which it is supposed that there is a tube closed at the bottom, open at the top, and of an altitude $= l$. There is besides supposed to be at the bottom of the tube a column of mercury whose height $= h$, and above this a column of air which under the pressure of the atmosphere would occupy the space y, and above this latter a new column of mercury which fills the rest of the tube. The tube is considered as placed under an exhausted receiver: the air included in the tube will then expand, and expel a portion of the column of mercury which presses upon it, until its elastic force is in equilibrio with the remainder of that superincumbent column. In this case x will be the distance between the bottom of the tube and that of the superior column of mercury after the expansion of the air.

563. Cor. 2. If any three of the four quantities h, l, x, y be given, the fourth may be found.

Ex. If the three first be given, the quantity of air left in before immersion is determined from the equation

$$y = \frac{h-x}{h} \cdot (l-x).$$

If $h = 30$, $l = 36$, and the mercury be depressed 10 inches, or $x = 20$,

$$y = \tfrac{1}{3} \cdot 16 = 5\tfrac{1}{3} \text{ inches.}$$

Prob. A given barometer containing some air is put under the receiver of an air-pump which contains n times as much as its barrel. Before the air is exhausted the mercury stands at an altitude a, and after m turns at an altitude b. Find the standard altitude, and the quantity of air in the tube at first.

Let $x =$ the standard altitude at first;

$\therefore x \cdot \left(\frac{n+1}{2n+1}\right)^m$ = the standard altitude after m turns (611).

Let $y =$ the space occupied by the air before immersion,

$l =$ the length of the tube;

$$\text{then } x : x - a :: l - a : y \quad (560),$$

$$\text{whence } xy = (x-a) \cdot (l-a).$$

Again, the elasticity (e) of the air occupying the space y is such as would support a column x of mercury, and the elasticity (e') of the air occupying $l-b$ is such as would support a column $x \cdot \left(\frac{n+1}{2n+1}\right)^m - b$ of mercury;

$$\therefore x : x \cdot \left(\frac{n+1}{2n+1}\right)^m - b :: l - b : y;$$

$$\text{whence } xy = \left\{x \cdot \left(\frac{n+1}{2n+1}\right)^m - b\right\} \cdot \{l-b\};$$

from which two equations x and y may be found.

Prob. Two barometers of the same given length are imperfectly filled; and the height of the mercury in each on two different days is observed. To determine the quantity of air contained in each.

Let l be the length of each tube,

a and $a' =$ the heights of the mercury in the first barometer on the two days respectively,

α and $\alpha' =$ the heights in the second.

Let $x =$ the length of the portion of the first tube occupied by the air originally,

and $y =$ that of the second;

and m and n the pressures of the atmosphere on the two days respectively measured by the altitudes of a perfect barometer; then (560) the depression below the standard altitude in the first on the first day will be $= \frac{mx}{l-a}$,

$$\left.\begin{aligned}\text{whence } \frac{mx}{l-a} + a &= m\\ \text{In a similar manner } \frac{my}{l-a'} + a' &= m\end{aligned}\right\} \text{(1).}$$

$$\left.\begin{aligned}\text{Also } \frac{nx}{l-a} + a &= n\\ \text{and } \frac{ny}{l-a'} + a' &= n\end{aligned}\right\} \text{(2).}$$

$$\text{Also from (1), } \frac{a}{a'} = \frac{1 - \dfrac{x}{l-a}}{1 - \dfrac{y}{l-a'}};$$

$$\text{whence } a - a' = \frac{ay}{l-a'} - \frac{a'x}{l-a} \quad (3);$$

and from (1) and (2),

$$y \cdot \left\{\frac{a-a'}{l-a'} - \frac{a-a'}{l-a'}\right\} = x \cdot \left\{\frac{a-a'}{l-a} - \frac{a-a'}{l-a'}\right\};$$

and from equations (3) and (4), x and y may be determined.

Prob. If two barometers, whose lengths are a and a' inches, contain b and b' inches of air respectively, and on account of some change in the weather, the former barometer falls one inch; to determine the depression in the latter, supposing a perfect barometer to stand at 30 inches before the depression.

Let x and x' be the altitudes after immersion when the standard altitude $= 30$; and when x becomes $x - 1$, let x' become $x' - y'$, and A be the new standard altitude;

$$\text{then (560)} \quad 30 : 30 - x :: a - x : b,$$

whence x may be found.

Again, $30 : A - (x - 1) ::$ elasticity of the air occupying b : elasticity of the air occupying $a - x + 1$,

$$:: a - x + 1 : b,$$

whence A may be found.

Similarly, in the other barometer x' may be found from the proportion

$$30 : 30 - x' :: a' - x' : b',$$

and y' the depression required from

$$30 : A - x' + y' :: a' - x' + y' : b'.$$

564. If the atmosphere consisted of particles repelling each other with forces varying inversely as any power of the distance between their centres, and the barometer be not entirely clear of air, an equation may be investigated for determining the height of the mercury after immersion.

If the repulsive force of the particles $\propto \frac{1}{d^n}$ (451), the compressing force $\propto D^{\frac{n+2}{3}}$. But the compressing force varies as the elasticity, and the density varies inversely as the space occupied; hence the elasticity varies inversely as the $\left(\frac{n+2}{3}\right)^{\text{th}}$ power of the space occupied;

$$\text{hence } E : E' :: ad^{\frac{n+2}{3}} : ab^{\frac{n+2}{3}} \text{ (figure Art. 560)};$$

$$\therefore ec : ed :: ad^{\frac{n+2}{3}} : ab^{\frac{n+2}{3}},$$

$$\text{or } h : h - x :: (l - x)^{\frac{n+2}{3}} : y^{\frac{n+2}{3}}.$$

Ex. If $n = 4$ or the repulsive force of the particles $\propto \frac{1}{d^n}$,

$$hy^2 = (h - x) \,.\, (l - x)^2;$$

whence x may be determined.

565. The altitude of a column of any other fluid of uniform density sustained by the air's pressure, is to the altitude of the mercury in the barometer as the specific gravity of mercury is to that of the other fluid.

For suppose two columns, one of mercury and the other of some other fluid supported by the air's pressure in two cylindrical tubes of equal diameters: their weights being equal to the weight of a column of air of the same base will be equal. Hence (13) $S \propto \frac{1}{M}$; and the base being the same, $M \propto$ the length;

$\therefore$ the altitude of the fluid : that of the mercury :: the specific gravity of mercury : that of the fluid.

566. Cor. 1. Hence the altitude at which water will be supported, may be ascertained.

If the mercury in the barometer be supposed to stand at 30 inches, and the specific gravities of water and mercury be as 1 : 13.568, the altitude of the water will = 34 feet very nearly.

To determine the height with accuracy in this and the following case, the specific gravity of the mercury in the barometer at the time of observation should be properly corrected.

567. Cor. 2. The height of an homogeneous atmosphere may also be determined.

If its specific gravity and that of mercury be in the ratio of 1 : 12000, and the barometer stand at 30 inches, the height required = 12000 × 30 inches = 10000 yards = $5\frac{1}{2}$ miles nearly.

568. If when the barometer and thermometer stand at given altitudes a and a', the weight of a cubical foot of air be W; to determine what will be the weight of a cubical foot when they stand at the altitudes a and a'; supposing that whilst a' increases one degree, nW is the quantity by which W is diminished.

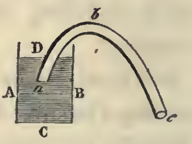
Let $W' =$ the weight required. Since in ascending 1°, W decreases by the quantity nW; therefore in ascending from a' to a', there will be a corresponding loss of weight represented by $nW.(a' - a')$. Hence a cubic foot of air which at the temperature a' had a weight W, will, at the temperature a', have a weight $= W - nW.(a' - a')$, the height of the barometer being $= a$. Hence W' and $W - nW.(a' - a')$ are the weights of the same magnitude of air at the same temperature a', the heights of the barometer being a and a. And since for the same temperature the weights of a cubic foot of air are in the direct ratio of the pressures,

$$W' : W - nW.(a' - a') :: a : a;$$

$$\therefore\ W' = \frac{a}{a}.[1 - n\cdot(a' - a')].W.$$

569. Def. A *Syphon* is a bent tube open at each end, having one of its legs longer than the other.

570. To explain the action of the syphon.



If the syphon abc be exhausted of air, and its shorter leg ba be immersed in a vessel of fluid ACB, the fluid will ascend through ab, and descend through the longer leg bc, till the surface of the fluid in the vessel is in the same horizontal plane with the end a.

The pressure of the atmosphere on the point D being unsupported, and the height of Db less than the standard altitude, the fluid rises into the tube, and having reached the highest point b, afterwards descends by its own weight through the leg bc and fills the tube. Now it is evident that the fluid in each leg will endeavour by its weight to descend; and this effort will be greater in bc than in ba; and as the pressure upwards of a column of atmosphere against the orifices c and a is the same, the effort of the fluid in the longer leg will prevail; or the fluid will continue to ascend through ab and descend through bc, till the surface of the fluid in ABC is depressed to a.

571. Cor. The form of the tube is indifferent: since the pressures of fluids depend upon the perpendicular altitudes only.

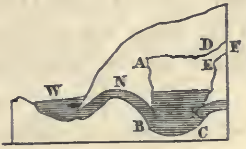
572. The syphon will cease to be effectual, if the altitude of ab be not considerably less than 34 feet in the case of

water, or than 30 inches in that of mercury; for the air contained in the fluid will have a tendency to escape and fill the vacuum in the syphon, and prevent the ascent of the fluid.

573. Def. A *Reciprocating Spring* is one which alternately flows and ceases to flow: or which discharges a greater quantity of water at one time than at another, after regular intervals.

574. On the principle of the syphon, the phænomena of reciprocating springs may be explained.

Let *ABCD* represent a cavern into which water is brought by the subterraneous passage *EF*; and suppose it to have an outlet *BNW* of a crooked form, whose highest point *N* is considerably above the bottom of the cavern, and thence sloping downwards into lower ground, and terminating in an open well at *W*, the dimensions of the canal *BN* being such as would discharge more water in a given time than is supplied by *EF*. This will produce a reciprocating spring at *W*; for when the cavern is filled higher than the point *N*, the canal *BNW* will act as a syphon, and by the supposition will discharge the water faster than *EF* supplies it; it will therefore run dry, and the well at *W* will cease to furnish water. After some time the cavern will again be filled to the height *N*, and the water will begin to flow at *W*.



If besides this supply, the well *W* receive water from a constant source, the spring will be reciprocating.

The situation and dimensions of this syphon-canal, and the supply of the feeder may be such that the efflux at *W* will be constant. But if the supply increase in a certain degree, a reciprocation will be produced at *W* with very short intervals: if the supply diminishes considerably, there will be another kind of reciprocation with great intervals, and great differences of water.

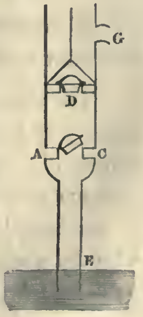
If the cavern have another simple outlet *K*, new varieties will be produced in the spring *W*, and *K* will afford a spring. If the entrance into the canal *K* be lower than *N*, and the supply through *EF* be no greater than *K* can discharge, there will be a constant spring from *K*, and *W* will have no water. But if the main feeder increases, yet not so much as to supply both *K* and *W*, the cavern will fill till the water rises above *N*, and *K* will continue running; but after *W* has begun to flow, and the water sinks below *K*, the stream from *K* will cease; and the cavern will be emptied by the syphon-canal, when *W* will cease flowing. The cavern will then begin to fill, and when the water is above *K*, *K* will flow; and soon after *W*; after which *K* will cease as before.

575. A *Syringe* consists of a barrel, furnished with a sucker made air-tight, and moveable by a rod.

Suppose the lower end of the syringe immersed in a fluid, and the sucker at its greatest depression; then will the air within the syringe between the lower extremity and the sucker be in its natural state. But when the sucker is elevated, the air below it occupying a greater space, its density, and therefore its elasticity, will be diminished (457). Whence the pressure of the external air being less counteracted will raise the fluid into the syringe; and by the subsequent depression of the sucker this fluid will be expelled.

576. Def. The term *Pump* is generally applied to a machine for raising water by means of the air's pressure.

577. The *Common Suction-Pump* consists of two hollow cylinders, which have the same axis and are joined in *AC*. The lower is partly immersed, perpendicularly in a spring or reservoir, and is called the *Suction-Tube;* the upper the *body of the pump*. At *AC* is a fixed sucker containing a valve which opens upwards, and is less than 34 feet from the surface of the water. In the body of the pump is a piston *D* made air-tight, moveable by a rod and handle, and



containing a valve opening upwards. And a spout *G* is placed at a small distance above the greatest elevation of *D*.

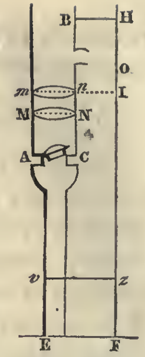
578. To explain the action of this pump.

Suppose the moveable piston *D* at its lowest depression, the cylinders free from water, and the air in its natural state. On raising this piston, the pressure of the air above it keeping its valve closed, the air in the lower cylinder *AE* forces open the valve at *AC*, and occupies a larger space, viz., between *E* the surface of the water and *D*; its elastic force therefore being diminished, and no longer able to sustain the pressure of the external air, this latter forces up a portion of the water into the cylinder *AE* to restore the equilibrium. This continues till the piston has reached its greatest elevation, when the valve at *AC* closes. In its subsequent descent, the air below *D* becoming condensed, keeps the valve at *AC* closed, and escapes by forcing open that at *D*, till the piston has reached its greatest depression. In the following turns a similar effect is produced, till at length the water rising in the cylinder forces open the valve at *AC*, and enters the body of the pump; when by the descent of *D*, the valve in *AC* is kept closed, and the water rises through that in *D*, which on re-ascending carries it forward, and throws it out at the spout *G*.

579. Cor. 1. The greatest height to which the water can be raised in the common pump by a single sucker is when the column is in equilibrio with the weight of the atmosphere, that is, between 32 and 36 feet.

580. Cor. 2. The quantity of water discharged in a given time is determined by considering that at each stroke of the piston a quantity is discharged equal to a cylinder whose base is a section of the pump, and altitude the play of the piston.

581. To determine the force necessary to overcome the resistance experienced by the piston in ascending.



Let h = the height HF of the surface of the water in the body of the pump above EF the level of the reservoir; and a^2 = the area of the section MN. Let h' = the height of the column of water equivalent to the pressure of the atmosphere; and suppose the piston in ascending to arrive at any position mn which corresponds to the height IF.

It is evident that the piston is acted upon downwards by the pressure of the atmosphere $= a^2 h'$, and by the pressure of the column $Bm = a^2 \times HI$; therefore the whole tendency of the piston to descend $= a^2 . (h' + HI)$.

But the piston is acted upon upwards by the pressure of the air on the external surface EF of the reservoir $= a^2 h'$; part of which is destroyed by the weight of the column of water having for its base mn, and height FI;

$\therefore$ the whole action upwards $= a^2 \times (h' - FI)$;

$$\text{whence } F = a^2 . (h' + HI) - a^2 . (h' - FI)$$
$$= a^2 . FH = a^2 h,$$

that is, the piston throughout its ascent is opposed by a force equal to the weight of a column of water having the same base as the piston, and an altitude equal to that of the surface of the water in the body of the pump above that in the reservoir. In order therefore to produce the upward motion of the piston, a force must be employed equal to that determined above, together with the weight of the piston and rod, and the resistance which the piston may experience in consequence of the friction against the inner surface of the tube.

When the piston begins to descend, it will descend by its own weight; the only resistance it meets with being friction and a slight impact against the water.

582. Cor. 1. If the water has not reached the piston, let its level be in vz. The under surface of the piston will be pressed by the internal rarefied air. But this air, together with the column of water Ev, is in equilibrio with the pressure

of the atmosphere a^2h'; and $\therefore$ its pressure $= a^2.(h' - Ev)$. And the pressure downwards $= a^2 h'$;

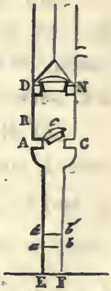
$$\therefore\ F = a^2 \times Ev.$$

Hence the force requisite to keep the piston in equilibrio increases as the water rises, and becomes constant and $= a^2 h$ as soon as the water reaches the constant level BH.

583. Cor. 2. If the weight of the piston be taken into the account, let this weight be equal to that of a column of water whose base is mn and height p, $= a^2 p$;

$$\therefore\ F = a^2.(Ev + p).$$

584. To determine the height to which the water will rise after one motion of the piston; the fixed sucker being placed at the junction of the suction-tube and body of the pump; supposing that after every elevation of the piston there is an equilibrium between the pressure of the atmosphere on the surface of the water in the reservoir, and the elastic force of the rarefied air between the piston and surface of the column of water in the tube, together with the weight of that column.



Let ab be the surface of the water in the suction-tube, after the first stroke of the piston: if the piston were for an instant stationary at D, the pressure of the atmosphere will balance Eb, and the elastic force of the air in Na.

Let AE the height of the suction-tube $= a$,

DR the play of the piston $= b$,

$h =$ the height of a column of water equivalent to the pressure of the atmosphere,

$y =$ the height of a column equivalent to the pressure of the air in Na,

$x = Ea$,

and R and $r =$ the radii of body and the suction-tube.

Then $x+y=h$,

$$\text{and } y=h\cdot\frac{AF}{Na}=h\cdot\frac{\pi r^2 a}{\pi R^2 b+\pi r^2\cdot(a-x)}$$

$$=\frac{hr^2 a}{R^2 b+r^2 a-r^2 x};$$

$$\text{whence } h=x+\frac{hr^2 a}{R^2 b+r^2 a-r^2 x};$$

$$\therefore hR^2 b+hr^2 a-hr^2 x=R^2 bx+r^2 ax-r^2 x^2+hr^2 a,$$

$$\text{or } x^2-\left(h+\frac{R^2}{r^2}\cdot b+a\right)\cdot x=-h\cdot\frac{R^2}{r^2}\cdot b,$$

$$\text{and } x^2-px=-hm'b,$$

$$\text{if } m \text{ be }=\frac{R^2}{r^2}, \text{ and } p=h+mb+a;$$

$$\therefore x=\tfrac{1}{2}\cdot\{p\pm\sqrt{p^2-4hmb}\},$$

$$\text{and } y=\tfrac{1}{2}\cdot\{2h-p\mp\sqrt{p^2-4hmb}\},$$

only one of which values will be applicable, viz. that which answers to the lower sign; since x and y must be less than h; and if the upper sign be used, x will be found greater than h.

585. Having given the height of the water raised, and that due to the pressure of the air in the pump after the first ascent of the piston; to determine them for the second, third, &c. ascents.

Let Ea' represent the height of the water after the second ascent, and let it $=x_1$,

and let $y_1=$ the height due to the elastic force of the air;

$$\text{then } x_1+y_1=h;$$

and $y_1=y\cdot\frac{Ab}{Na'}$, since the air which occupied Ca now occupies Na';

$$\therefore y_1=\frac{yr^2\cdot(a-x)}{R^2 b+r^2\cdot(a-x_1)}=\frac{y\cdot(a-x)}{mb+a-x_1};$$

$$\text{whence } h=x_1+\frac{y\cdot(a-x)}{mb+a-x_1},$$

and $\therefore\ x_1 = \frac{1}{2}\cdot\{p - \sqrt{p^2 - 4hmb - 4x\cdot(h+a-x)}\}$,

and $y_1 = \frac{1}{2}\cdot\{2h - p + \sqrt{p^2 - 4hmb - 4x\cdot(h+a-x)}\}$.

From these are deduced values of x_2, y_2, x_3, y_3, &c.

$$x_2 = \frac{1}{2}\cdot\{p - \sqrt{p^2 - 4hmb - 4x_1\cdot(h+a-x_1)}\},$$

$$y_2 = \frac{1}{2}\cdot\{2h - p + \sqrt{p^2 - 4hmb - 4x_1\cdot(h+a-x_1)}\},$$

and so on. Whence if x_n be taken to represent the height of the water after $(n+1)$ ascents,

$$x_n = \frac{1}{2}\cdot\{p - \sqrt{p^2 - 4hmb - 4x_{n-1}\cdot(h+a-x_{n-1})}\},$$

and $y_n = \frac{1}{2}\cdot\{2h - p + \sqrt{p^2 - 4hmb - 4x_{n-1}\cdot(h+a-x_{n-1})}\}$.

586. Cor. 1. Hence may be determined the height to which the water can rise after any given number of ascents of the piston, and the elastic force of the air in the suction-tube.

587. Cor. 2. Knowing the elevation due to each particular stroke, the differences of those elevations, and the successive differences in the elastic force of the remaining air, may be known.

588. Cor. 3. If the weight of the valve c be not considered, it is evident that after a certain number of strokes a vacuum will be produced in the suction-tube, provided it be equal to, or not greater than the height due to the pressure of the atmosphere, that is, if a be not greater than h.

For in this case, $x_n = x_{n-1}$,

and $\therefore\ x_{n-1} = \frac{1}{2}\{p - \sqrt{p^2 - 4hmb - 4x_{n-1}\cdot(h+a-x_{n-1})}$),

whence $x_{n-1} = h$, the greatest height of the column of water in the tube. If therefore the length of the suction-tube do not exceed the height due to the pressure of the atmosphere, the water will continue to ascend in it after every stroke of the piston, till at length it will pass into the body of the pump.

But if the altitude of AF be greater than h, the water will continue to ascend without ever reaching its maximum height. For in this case, an actual vacuum cannot be produced; and as $x_n + y_n = h$, and y_n can never become $= 0$; $\therefore$ x_n can never $= h$*. But the successive values of y continually decreasing, the corresponding values of x will continually increase.

589. Cor. 4. If the weight of the valve c be taken into the account, a column of water must be added equal to the additional pressure to be overcome. Let $l =$ the height of of this column, then

$$x + y + l = h;$$

$$\text{and } \therefore\ x + y = h - l = h'.$$

If therefore this value of h' be substituted for h, the preceding equations are applicable.

590. Cor. 5. In the preceding cases, the moveable piston has been supposed to descend to AC. If it does not, it may happen that the water may not reach AC, though AC be less than 34 feet from the surface of the water in the reservoir.

After the first elevation of the moveable piston to its greatest altitude, c being closed, the elastic force of the air between DN and AC is $(h - x)$, and its magnitude $\pi b R^2$. If in descending, the piston describes a space b' less than b, so as to stop at a distance $b - b'$ from AC, this magnitude becomes $(b - b') . \pi R^2$; $\therefore$ the elastic force is $(h - x) . \dfrac{b}{b - b'}$. Now in order that the pressure upwards may open the valve, this must exceed the elastic force of the atmosphere;

$$\therefore\ (h - x) . \frac{b}{b - b'} > h,$$

$$\text{or } (h - x) . b > h . (b - b');$$

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* Hence it appears that it is not *strictly* true, that water will ascend in the suction-tube to a height equal that of a column equivalent to the pressure of the atmosphere. This is a limit to which it approximates, but does not reach in a finite time.

$$\therefore\ bx < hb',$$

$$\text{or } \frac{x}{h} < \frac{b'}{b}.$$

If $\therefore\ \frac{b'}{b}$ be less than $\frac{x}{h}$, the valve DN will not open; there will therefore be the same quantity of air between AC and the sucker: which, when the piston has reached its highest elevation, will have the same elastic force as that between AC and $a'b'$; and therefore c being equally pressed on both sides, will remain unmoved, and the water will not ascend.

591. If the fixed sucker be placed at the surface of the water; to determine the ascent of the water in the suction-tube.

Let Ea, Ea' be the successive heights to which the water rises; then after the first ascent of the piston,

$$x + y = h,$$

$$\text{and } y = \frac{ha}{mb + a - x};$$

$$\text{whence } x = \tfrac{1}{2}.\{p - \sqrt{p^2 - 4hmb}\},$$

$$\text{and } y = \tfrac{1}{2}.\{2h - p + \sqrt{p^2 - 4hmb}\},$$

which equations are the same as were determined for the first ascent of the piston (584).

Now at the beginning of the second ascent, the air in Ab is in its natural state, which after the second ascent, being diffused through the space Db', the height due to its elastic force will be

$$y_1 = h.\frac{Ab}{Db'} = \frac{h.(a - x)}{mb + a - x},$$

$$\text{and } x_1 + y_1 = h,$$

$$\text{whence } x_1 = \tfrac{1}{2}.\{p - \sqrt{p^2 - 4hmb - 4hx}\},$$

$$\text{and } y_1 = \tfrac{1}{2}.\{2h - p + \sqrt{p^2 - 4hmb - 4hx}\}.$$

In the same manner $x_2 = \frac{1}{2}.\{p - \sqrt{p^2 - 4hmb - 4hx_1}\}$,

$$\text{and } y_2 = \tfrac{1}{2}.\{2h - p + \sqrt{p^2 - 4hmb - 4hx_1}\};$$

$$x_n = \tfrac{1}{2}.\{p - \sqrt{p^2 - 4hmb - 4hx_{n-1}}\},$$

$$y_n = \tfrac{1}{2}.\{2h - p + \sqrt{p^2 - 4hmb - 4hx_{n-1}}\}.$$

592. Cor. 1. If the water be supposed to stop after $(n+1)$ ascents of the piston, then $x_n = x_{n-1}$;

$$\text{and } \therefore x_{n-1} = \tfrac{1}{2}.\{p - \sqrt{p^2 - 4hmb - 4hx_{n-1}}\},$$

$$\text{whence } x_{n-1} = \tfrac{1}{2}.\{a + mb \pm \sqrt{(a+mb)^2 - 4hmb}\}.$$

Hence therefore there are two altitudes at which the water may stop in its ascent, if $(a+mb)^2$ is equal to or greater than $4hmb$. In the former case the two values of x_{n-1} are equal, that is, there will be only one altitude $= \frac{1}{2}.(a+mb)$, at which the water will stop. In the latter case there are two which may be ascertained.

If $4hmb$ be greater than $(a+mb)^2$, the water will not stop.

Ex. 1. If $h = 32$ feet, $a = 20$, $b = 4$, and $m = 1$ or the suction-tube and body of the pump be of the same diameter,

$$x_{n-1} = \tfrac{1}{2}.\{20 + 4 \pm \sqrt{(24)^2 - 4.1.32.4} = \tfrac{1}{2}.\{24 \pm \sqrt{64}\}$$
$$= 16 \text{ or } 8.$$

Ex 2. If $h = 32$ feet, $a = 25$, $b = 2$, and $m = 4$,

$$x_{n-1} = \tfrac{1}{2}.\{25 + 8 \pm \sqrt{(33)^2 - 4.32.4.2} = \tfrac{1}{2}.\{33 \pm \sqrt{65}\}.$$

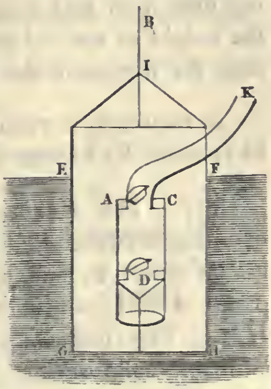
593. Cor. 2. If $m = 1$, or the tubes have the same diameter,

$$x_{n-1} = \tfrac{1}{2}.\{a + b \pm \sqrt{(a+b)^2 - 4hb}\},$$

which is imaginary, if $(a+b)^2$ is less than $4hb$, or b greater than $\frac{(a+b)^2}{4h}$.

In order therefore that this pump may produce its effect, the play of the piston must be greater than the square of its greatest altitude above the surface of the water in the reservoir divided by four times the height due to the pressure of the atmosphere.

594. The *Lifting-Pump* consists of a hollow cylinder, the body of which is immersed in the reservoir whose surface is *EF*. It is furnished with a moveable piston *D*, which entering below lifts the water up, and is moveable by means of a frame *BIHG* which is made to ascend and descend by a handle. The piston is furnished with a valve opening upwards. A little below the surface of the water is a fixed sucker *AC* with a valve opening upwards. At *AC* is inserted the ascending tube *AK*, into which the water is raised.



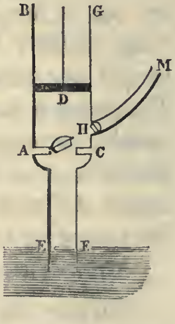
595. To explain the action of this pump.

Suppose the piston *D* at its lowest point, the body of the pump filled with water to the level of the reservoir, and the valves closed. As *D* ascends, its valve will be kept closed, and the pressure of the ascending water will open that at *AC*, and pass into the tube *ACK*. During this ascent, the water in the reservoir will ascend into the tube after *D*, there being no force to counteract it, and the tube will become filled. When the piston has reached its greatest elevation, the valve at *AC* will close by its own gravity· and upon the descent of *D* will be kept closed by the incumbent water, whilst the pressure of that below *D* will open its valve, and pass into the space between *D* and *AC*; when *D* having reached its lowest point, its valve closes; and upon the subsequent ascent of the sucker, the water will be raised through *AC* into the ascending tube: and by repeating the operation may be raised to the required altitude.

596. To determine the force necessary to overcome the resistance experienced by the piston in its ascent.

The pressure upon the piston is equal to the weight of a column of water whose base is equal to that of the piston, and altitude the distance of the surface of the water in the ascending tube from that in the reservoir. To this must be added the weight of the rod and frame, and the resistance arising from the friction of the sucker in the barrel.

In returning, it would descend by its own weight; but will be retarded by friction and a slight impact against the water.



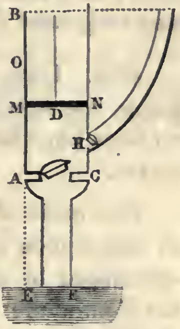
597. The *Forcing-Pump* consists of of a suction-tube *AEFC* partly immersed in the reservoir, and of the body of the pump *ABGC*, and of the ascending tube *HM*. The body is furnished with a moveable solid sucker *D*, made air-tight. And at *AC* and *H* are fixed suckers with valves opening upwards.

598. To explain the action of this pump.

Suppose the sucker *D* at its greatest depression; the valves closed, and the air in its natural state. Upon the ascent of *D*, the air in *ACD* occupying a greater space, its elasticity will be diminished, and consequently the greater elasticity of the air in *AF* will open the valve at *AC*, whilst that at *H* is kept closed by the elasticity of the external air: water therefore will rise in the suction-tube. On the descent of *D* from its greatest elevation, the elasticity of the air in the body of the pump will keep the valve *AC* closed, and open that at *H*, whence air will escape. By subsequent ascents of the piston, the air will be expelled, and water rise into the body. The descending piston will then press the water through the valve at *H*, which will close, and prevent its return into the body of the pump. *D* therefore ascending again, the space left void will be filled by water pressing through the valve *AC*: and this upon the next ascent of *D* is forced into the ascending tube; and thus by the ascents and descents of D, water may be raised to the required height.

599. Cor. In this pump, D must not ascend higher than about 32 feet from the surface of the water in the reservoir.

600. To determine the force necessary to overcome the resistance experienced by the piston.



Let $h =$ the height of a column of water equivalent to the pressure of the atmosphere, and EB the height to which the water is forced. Let MN be any position of the piston D whose area $= A$, and the weight of the piston and its appendages $= P$. Let $X =$ the force necessary to push the piston upwards during the suction, friction not being considered, and $Y =$ that employed to force it down.

When the piston ascends, and H is closed

$$X = P + Ah - A.(h - ME)$$
$$= P + A.ME.$$

Let the sucker be in the same position in its descent, and therefore AC closed, and H open,

$$Y = Ah + A.MB - (Ah + P)$$
$$= A.MB - P.$$

Hence $X + Y = A.EB$; or the whole force exerted, in the case of equilibrium is equal to the weight of a column of water whose base is equal to that of the piston, and altitude the distance between the surface of the water and the point to which it is to be raised.

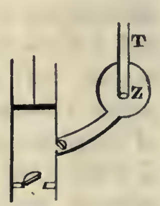
601. Cor. 1. In this pump the effort is divided into two parts, one opposed to the suction, and the other to the forcing; whereby an advantage is gained over the other pumps

where the whole force is exerted at once whilst the water is raised.

602. Cor. 2. In order to have the force applied uniform, let $X=Y$;

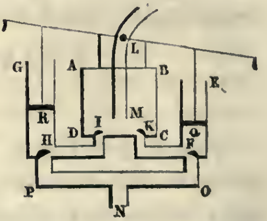
$$\therefore\ P+A\,.\,ME=A\,.\,MB-P;$$
$$\therefore\ P=\tfrac{1}{2}A\,.(MB-ME).$$

The piston therefore must play in such a manner that *MB* may be greater than *ME*.



603. Cor. 3. In the common forcing-pump, the stream is intermitting; for there is no force impelling it during the return of the sucker.

One mode of remedying this, is by making an interruption in the ascending tube, which is surrounded by an air-vessel *T*; in which, when the water has rizen above *Z*, the air above it is compressed, and by its elasticity forces the water up through *Z*; the orifice of which is narrower than that of the tube, and therefore the quantity of water introduced during the descent of the piston will supply its discharge for the whole time of the stroke, producing a continued stream.

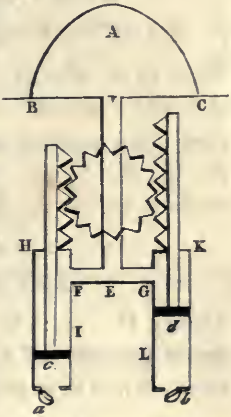


604. The *Fire-Engine* consists of a large receiver *ABCD*, called the *Air-Vessel*, into which water is driven by two forcing-pumps *EF*, *GH*, (whose pistons are *Q* and *R*), communicating with its lower extremities at *I* and *K*, through two valves opening inwards. From the receiver proceeds a tube *ML* through which the water is thrown, and directed to any point by means of a pipe moveable about the extremity *L*. The pumps are worked by a lever, so that whilst one piston descends the other ascends. The pumps communicate with a reservoir of water at *N*.

605. To explain the action of this engine.

The tube *N* being immersed in the reservoir, and the piston *R* drawn up, the pump *GH* becomes filled; and the descent of the piston *R* will as in the forcing-pump (598) keep the valve *H* close, and cause the water to pass into the air-vessel by the valve *I*, whilst by the weight of the water in the air-vessel, the valve *K* will be kept shut. In the same manner, when *R* ascends, *Q* descending will force the water through *K* into the air-vessel. By this means the air above the surface of the water becoming greatly compressed will by its elasticity force the water to ascend through *ML*, and to issue with a great velocity from the pipe.

606. The *Air-Pump* consists of a glass vessel *A*, called the *receiver*, closely fitted to a horizontal table *BC*, and communicating by means of a vertical pipe *DE*, branching at its lower extremity *E* in opposite directions *EF*, *EG*, with two cylindrical brass vessels *HI*, *KL*, called *barrels*. Each barrel is furnished, 1. with a valve as *a* or *b* placed at its lower extremity and opening outwards; 2. with a sucker *c* or *d*, air-tight and moveable by a rod and rack-work, in such a manner that whilst one is elevated, the other is depressed; 3. with an orifice as *F* or *G* near its upper extremity, by which it communicates with the receiver.



607. To explain the action of the air-pump.

Suppose it filled with common air, the valves closed, and one sucker *d* at its greatest elevation. Upon the descent of *d* below the orifice of communication, the air in *db* occupying a less space than before, its elasticity will be increased, and will therefore open the valve *b*; and by the whole descent of d a barrel full of air will be expelled. Next suppose d having reached its greatest depression to ascend; the valve at b (previously closed by a spring) will be kept closed by the pressure of the external air, that in bd being considerably exhausted; and upon the ascent of d above G, the barrel will be replenished out of the receiver. In this manner the operation is continued, c ascending whilst d descends, and *vice versâ*. And a barrel full of air being exhausted by each turn of the handle, by repeated turns the air may be exhausted at pleasure.

608. The quantities of air remaining in the machine after successive turns decrease in geometrical progression.

Let $A =$ the capacity of the receiver and pipes,
$B =$ the capacity of either barrel;
$\therefore\ A + 2B =$ the capacity of the machine.

Now $Q \propto M \times D$; and if D be given, $Q \propto M$; hence $A + 2B : A + B ::$ the quantity in the machine before any turn : the quantity remaining after that turn. Let therefore P, Q, R, S, &c. represent the quantities of air in the machine before the first, second, third, &c. turns respectively;

$$\therefore\ A + 2B : A + B :: P : Q,$$
$$\text{and } A + 2B : A + B :: Q : R, \text{ \&c.}$$

whence $P : Q :: Q : R :: R : S$, &c. or P, Q, R, S, &c. are in geometrical progression; and it is a decreasing one, since $A + 2B$ is greater than $A + B$.

609. Cor. Hence the air can never be wholly exhausted.

For no assignable term of a geometrical progression is evanescent.

610. The quantities of air exhausted by successive turns decrease in geometrical progression.

Retaining the same notation, it may be proved as before that $P : Q :: Q : R$;

$$\therefore\ P : Q :: P - Q : Q - R.$$

In the same manner it may be shewn that

$$Q : R :: Q-R : R-S.$$

And since the first ratios are equal,

$$P-Q : Q-R :: Q-R : R-S.$$

In the same manner, $Q-R : R-S :: R-S : S-T$.

Now $P-Q$, $Q-R$, $R-S$, $S-T$, &c. are the quantities exhausted by the first, second, third, &c. turns; which are therefore in geometrical progression; and this is a decreasing one, since (608) P is greater than Q.

611. To determine the quantity of air remaining in the machine after any number of turns.

It has been shewn (608) that $A+2B : A+B :: P : Q$,
and $A+2B : A+B :: Q : R$,

and so on for n turns; whence compounding the proportions,

$$(A+2B)^n : (A+B)^n :: P : \text{quantity after } n \text{ turns},$$

$$\text{which } \therefore \text{ is} = P \, . \left(\frac{A+B}{A+2B}\right)^n.$$

612. Cor. To determine the quantity exhausted by any number of turns.

The quantity at first being $= P$, and the quantity remaining after n turns $= P \, . \left(\frac{A+B}{A+2B}\right)^n$;

$$\therefore \text{ the quantity exhausted} = P - P \, . \left(\frac{A+B}{A+2B}\right)^n$$

$$= P \, . \frac{(A+2B)^n - (A+B)^n}{(A+2B)^n}.$$

613. To find the density of the air in the machine after any number of turns.

Since $Q \propto D$ when M is given, and the quantities at first and after n turns are in the ratio of $(A+2B)^n : (A+B)^n$; if D and $d =$ the densities at first and after n turns;

$$\therefore (A+2B)^n : (A+B)^n :: D : d,$$

$$\text{or } d = D \,.\left(\frac{A+B}{A+2B}\right)^n.$$

614. Cor. 1. The density of the air in the machine after successive turns decreases in geometrical progression.

For $D \propto Q$ when M is given; and the quantity remaining decreases in geometrical progression.

615. Cor. 2. To determine the number of turns by which the air will be reduced to a given density.

$$\text{Since } d = D \,.\left(\frac{A+B}{A+2B}\right)^n;$$

$$\therefore \log. d - \log. D = n \,. \{\log. (A+B) - \log. (A+2B)\},$$

$$\text{and } n = \frac{\log. d - \log. D}{\log.(A+B) - \log.(A+2B)}$$

$$= \frac{\log. d}{\log.(A+B) - \log.(A+2B)}, \text{ if the original density} = 1.$$

But since d is here a fraction, let $\frac{1}{d} = R$, where $R =$ the whole number representing the rarefaction;

$$\therefore \log. R = -\log. d,$$

$$\text{and } n = \frac{\log. R}{\log. (A+2B) - \log. (A+B)}.$$

616. Cor. 3. Since $\log. \left(\frac{A+B}{A+2B}\right) = \frac{\log. d - \log. D}{n}$, when the number of turns, and the densities are given, the ratio of $A+2B : A+B$; and therefore of $A : B$ may be determined.

Prob. A body when placed under the receiver of a given air-pump weighs a ounces, and after n turns weighs b ounces.

Required the weight of the body in vacuo; and supposing the specific gravity of the body known, determine the density of the air in the receiver at first.

Let W = the weight in vacuo,

S = the specific gravity of the solid,

and s = the density of the air at first;

and $p : 1$ = the ratio of the capacities of the barrel and receiver;

$$\therefore \text{ the density after } n \text{ turns} = \left(\frac{p+1}{2p+1}\right)^n . s = ms,$$

$$\text{if } m \text{ be put} = \left(\frac{p+1}{2p+1}\right)^n.$$

Now the weight lost $= \dfrac{s}{S} . W$ (110);

$$\therefore\ W - \frac{s}{S} . W = a,$$

$$\text{and } W - m . \frac{s}{S} . W = b;$$

whence $W . \dfrac{s}{S} . (1 - m) = b - a$,

$$\text{and } W . \frac{s}{S} = \frac{b-a}{1-m}.$$

$$\text{Hence } W - \frac{b-a}{1-m} = a,$$

$$\text{and } W = \frac{b - ma}{1 - m} = \frac{(2p+1)^n . b - (p+1)^n . a}{(2p+1)^n - (p+1)^n}.$$

$$\text{Also the weight lost} = \frac{b-a}{1-m};$$

$$\therefore\ \frac{b-a}{1-m} : \frac{b-ma}{1-m} :: s : S;$$

$$\therefore\ s = S . \frac{b-a}{b-ma} = S . \frac{(2p+1)^n . (b-a)}{(2p+1)^n b - (p+1)^n . a}.$$

617. The *Gage* of an air-pump is a vertical glass tube communicating with the receiver by means of a horizontal tube into which its upper end is fixed, the lower being immersed in a basin of mercury; and from the point of immersion it is graduated up to 31 inches.

618. As the air is exhausted the mercury will rise in the gage; and its defects from the standard altitude after each successive turn decrease in geometrical progression.

For the density of the air within the receiver and gage and therefore its compressing force on the mercury decreasing at every turn, and the compressing force of the external air upon the mercury in the basin remaining the same; the mercury will rise in the gage, till its weight together with the remaining elasticity of the air in the receiver is in equilibrio with the whole pressure of the atmosphere. If then all the air were exhausted, the mercury would rise to the standard altitude. Now since the compressing force of the air in the receiver prevents the mercury from rising to the standard altitude, it must equal the weight of a column of mercury which is equal to the defect; and therefore the defect being as the compressing force must vary as the density (455), and therefore decrease in geometrical progression (614).

619. Cor. 1. Hence the expansion of the air in the receiver may be determined.

Let $M =$ the magnitude of the receiver,

$h =$ the height to which the mercury rises in the gage, in consequence of the rarefaction,

$H =$ the height of the mercury in the barometer.

Then if the air in the receiver were pressed by the force H, its magnitude would be diminished; let it $= m$; and suppose $M = n \, . \, m$; n will be the measure of the expansion of the air.

Now its elastic force in this state of expansion $= H . \dfrac{m}{M} = \dfrac{H}{n}$;

and the pressure of the atmosphere $H = h + \frac{H}{n}$;

$$\therefore\ n = \frac{H}{H - h}.$$

620. Cor. 2. The weight of the mercury in the gage is the excess of the weight of the atmosphere above the elasticity of the included air.

621. Cor. 3. If the tube of a common barometer be placed under a receiver of sufficient length, and the air be exhausted, the mercury in the tube will descend, whilst that in the gage will rise; and the sum of their heights will be always the same, viz. the standard altitude: the height of the mercury in the receiver being the effect of the elasticity of the remaining air, and the height of that in the gage the unbalanced pressure of the atmosphere.

622. Cor. 4. The ascents of the mercury in the gage at each successive turn decrease in geometrical progression.

Since the defects of the mercury from the standard altitude decrease in geometrical progression, and the differences of these defects are the successive ascents; these ascents therefore decrease in geometrical progression.

Prob. Given the altitude of the mercury in the gage of an air-pump ($= h$) after n ascents of the piston; the standard altitude $= H$; to compare the capacities of the receiver and barrel.

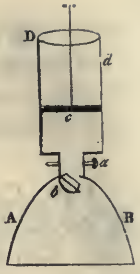
The defect from the standard altitude $= H - h$;

$$\therefore\ H : H - h :: \text{density at first} : \text{density at last}$$

$$:: (A + 2B)^n : (A + B)^n;$$

$$\text{and } H^{\frac{1}{n}} : (H - h)^{\frac{1}{n}} :: A + 2B : A + B;$$

$$\therefore\ H^{\frac{1}{n}} - (H - h)^{\frac{1}{n}} : 2 . (H - h)^{\frac{1}{n}} - H^{\frac{1}{n}} :: B : A.$$



623. The *Condenser* consists of a vessel AB, into the upper surface of which is inserted a stop-cock a communicating with a cylindrical barrel DE, which is furnished with a solid sucker c, air-tight, and moveable by a rod and handle; and with an aperture d at a small distance below the greatest elevation of the sucker. At b is a valve opening downwards, which is closed by a spring.

624. To explain the action of the condenser.

Suppose the sucker at its greatest elevation, the valve b closed, and the air in every part of the machine in its natural state. When the sucker descends below d, the air in cb occupying a less space than before, its elasticity will be increased, and therefore being greater than that in the receiver, will open the valve b; thus by the continued descent of the sucker a barrel full of common air will be impelled into the receiver. When the sucker has reached its greatest depression, the air below it being reduced to the same density, the valve b will be closed by the spring; and upon the ascent of the sucker, the quantity of air in cb being inconsiderable, it will be kept closed by the pressure of the air in the receiver. When c rises above the orifice d, the barrel will be replenished with common air. Thus the operation may be continued.

625. Cor. 1. The quantities of air introduced by each descent are equal.

626. Cor. 2. The elastic force of the air in the receiver being increased by every stroke, it is manifest that the descent of the sucker after each turn must be increased before the elasticity of the air above the valve b be sufficient to force it open.

627. Cor. 3. The quantities of air in the receiver after successive descents are in arithmetic progression.

For if A = the quantity in the receiver at first, and B the quantity contained in the barrel; $A + B$, $A + 2B$,

$A+3B$, &c. are the quantities after the first, second, third, &c. descents.

628. **Cor.** 4. The quantity in the receiver after n descents $= A+nB$.

629. **Cor.** 5. Since $Q \propto D$ when M is given; if D and $d =$ the densities at first and after n descents,

$$A : A+nB :: D : d;$$

$$\therefore\ d = D.\frac{A+nB}{A} = D.\left(1+\frac{nB}{A}\right).$$

630. **Cor.** 6. The density increases in arithmetic progression.

Prob. Given the capacity of the barrel, neck and receiver of a condenser; to determine how many descents of the sucker will be necessary to make the density of the air in the receiver a max.

Let b, n, r represent the capacities of the barrel, neck, and receiver respectively, and let $b+n=a$, and $n+r=a'$.

If the density of the air in its natural state $=1$, $\frac{a+r}{a'} =$ the density after the first descent (629).

And $\left(a\times 1+r.\frac{a+r}{a'}\right).\frac{1}{a'} = \frac{a}{a'}+\frac{r}{a'}.\frac{a+r}{a'} =$ the density after the second.

Also $\left\{a\times 1+r.\left(\frac{a}{a'}+\frac{r}{a'}\times\frac{a+r}{a'}\right)\right\}.\frac{1}{a'} = \frac{a}{a'}.\left\{1+\frac{r}{a'}\right\}$

$+\frac{r^2}{a'^2}.\frac{a+r}{a'} =$ the density after the third.

And $\left\{a\times 1+r\left[\frac{a}{a'}.\left(1+\frac{r}{a'}\right)+\frac{r^2}{a'^2}.\frac{a+r}{a'}\right]\right\}.\frac{1}{a'}$

$=\frac{a}{a'}\times\left\{1+\frac{r}{a'}+\frac{r^2}{a'^2}\right\}+\frac{r^3}{a'^3}.\frac{a+r}{a'} =$ the density after the fourth.

Hence the density after the x^{th} descent

$$=\frac{a}{a'}\cdot\left\{1+\frac{r}{a'}+\frac{r^2}{a'^2}+\ldots\ldots+\frac{r^{x-2}}{a'^{x-2}}\right\}+\frac{r^{x-1}}{a'^{x-1}}\times\frac{a+r}{a'}$$

$$=\frac{a}{a'}\times\frac{\left(\frac{r}{a'}\right)^{x-1}-1}{\frac{r}{a'}-1}+\left(\frac{r}{a'}\right)^{x-1}\times\frac{a+r}{a'}.$$

But the ultimate density $=\frac{b+n}{n}$, since $D\propto\frac{1}{M}$;

whence

$$\frac{b+n}{n+r}\cdot\frac{1-\left(\frac{r}{n+r}\right)^{x-1}}{\frac{n}{n+r}}+\frac{b+n+r}{n+r}\cdot\left(\frac{r}{n+r}\right)^{x-1}=\frac{b+n}{n},$$

or $\frac{b+n}{n}\cdot\left\{1-\left(\frac{r}{n+r}\right)^{x-1}\right\}+\frac{b+n+r}{n+r}\cdot\left(\frac{r}{n+r}\right)^{x-1}=\frac{b+n}{n}$;

$$\text{whence }\left\{\frac{b+n+r}{n+r}-\frac{b+n}{n}\right\}\cdot\left(\frac{r}{n+r}\right)^{x-1}=0,$$

$$\text{and }\therefore\ \left(\frac{r}{n+r}\right)^{x-1}=0,$$

which can only happen when the number of descents of the sucker is infinite; since $\frac{r}{n+r}$ is a proper fraction.

Prob. A receiver whose magnitude is c, has two barrels connected with it; one of which, whose magnitude is a, condenses; the other whose magnitude is b, exhausts; and they take their strokes alternately; to determine the effect after an infinite number of strokes.

Let $s=$ the density of the atmosphere, and s', s'', s''', &c. the densities after the first, second, third, &c. strokes;

$$\therefore\ c:c+a::s:s';$$

$$\therefore\ s' = \frac{c+a}{c} \cdot s;$$

$$\text{and } c+b : c :: \frac{c+a}{c} \cdot s : s'';$$

$$\therefore\ s'' = \frac{c+a}{c+b} \cdot s.$$

In the same manner $s''' = s \cdot \left(\frac{c+a}{c+b} + \frac{a}{c}\right)$;

$$s^{\text{IV}} = s \cdot \left(\frac{c \cdot (c+a)}{(c+b)^2} + \frac{a}{c+b}\right);$$

$$s^{\text{V}} = s \cdot \left(\frac{c \cdot (c+a)}{(c+b)^2} + \frac{ca}{c+b} + \frac{a}{c}\right);$$

$$s^{\text{VI}} = s \cdot \left(\frac{c^2 \cdot (c+a)}{(c+b)^3} + \frac{ca}{(c+b)^2} + \frac{a}{c+b}\right).$$

It is therefore evident that after an infinite number of strokes, b working the last, that the density will be

$$= s \cdot \left(\frac{a}{c+b} + \frac{ca}{(c+b)^2} + \frac{c^2 a}{(c+b)^3} + \&\text{c. } in\ inf.\right),$$

since the last term $\frac{c^n \cdot (c+a)}{(c+b)^{n+1}}$ will be $= 0$, when n is infinite.

$$\text{Hence the density} = s \cdot \frac{a}{b}.$$

But if a work the last, the ultimate density will be

$$= s \cdot \left(\frac{a}{b} + \frac{a}{c}\right) = sa \cdot \frac{b+c}{bc}.$$

Prob. The barrels of an air-pump communicate with the receiver of a condenser which is of the same magnitude as that of the pump. The density of air in the condenser is $(p+1)$ times that in the pump, which is in its natural state; and a barometer-tube having the basin of mercury in the condenser, has its upper end, which is *open*, in the pump.

A piston of the same diameter as the tube, and whose weight is equal to the weight of a column of mercury of the standard altitude, is placed in the tube and suffered to descend. Find its place, when at rest at first, and after n turns; the ratio of either receiver to a barrel being $r-1 : 1$.

Let ma = the length of the tube,
a = the standard altitude.

Before the piston is inserted, the mercury stands at an altitude pa; and when it is at rest at first, the pressure on the air between it and the mercury being doubled, the space occupied is halved, or is $\frac{1}{2}.(m-p).a$; and the elasticity being in consequence doubled, the mercury sinks through a space a. Hence the altitude at which the piston rests at first is

$$\left\{\tfrac{1}{2}.(m-p)+p-1\right\}.a=\left\{\tfrac{1}{2}.(m+p)-1\right\}.a.$$

After n turns the diminution of density in the pump is $1-\left(\frac{r}{r+1}\right)^n$, if 1 represent the density of the air at first.

And the increase of density in the condenser is

$$\frac{1}{r-1}.\left\{1+\frac{r}{r+1}+\left(\frac{r}{r+1}\right)^2\ldots\ldots\ldots+\left(\frac{r}{r+1}\right)^{n-1}\right\}$$

$$=\frac{1}{r-1}.\frac{1-\left(\frac{r}{r+1}\right)^n}{\frac{1}{r+1}}=\frac{r+1}{r-1}.\left\{1-\left(\frac{r}{r+1}\right)^n\right\};$$

therefore the increase of the altitude of the mercury from both causes $=a.\frac{2r}{r-1}.\left\{1-\left(\frac{r}{r+1}\right)^n\right\}$.

Also if s = the space occupied by the air between the mercury and piston,

$$s : \tfrac{1}{2}(m-p).a :: 2 : 1+\left(\frac{r}{r+1}\right)^n;$$

therefore the increase of space occupied by the air

$$=\tfrac{1}{2}\cdot(m-p)\cdot a\cdot\frac{1-\left(\frac{r}{r+1}\right)^n}{1+\left(\frac{r}{r+1}\right)^n};$$

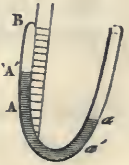
therefore the ascent of the piston upon the whole is

$$=a\cdot\left\{\frac{2r}{r-1}+\frac{\frac{1}{2}(m-p)}{1+\left(\frac{r}{r+1}\right)^n}\right\}\cdot\left\{1-\left(\frac{r}{r+1}\right)^n\right\}.$$

631. The *Gage* of a condenser is a cylindrical glass tube consisting of two branches, one of which is hermetically sealed, and the other open. The extremity of the sealed branch is occupied by a certain quantity of dry air, which is confined by a portion of mercury.

It is evident that as the air is condensed in the receiver, its pressure on the surface of the mercury in the tube will make it advance in the sealed branch, and that this will be opposed by the elasticity of the air in that branch. As the same quantity of air therefore will occupy successively less and less spaces which may be measured, the ratio between the densities of the air in the receiver at first and after any given number of turns may be compared.

632. To determine the density of the air in the receiver.



Suppose at the commencement of an experiment the mercury stood at A in the sealed branch, and at a in the open one, so that the excess of the height of the first column above that of the second $=h$. Let $H=$ the height of the mercury in the barometer; then the elastic force of the air in $AB=H-h$. Let M represent its magnitude determined by the number of divisions on the scale.

After a number of descents of the sucker, let the mercury rise to A' in the sealed branch, and descend to a' in the other;

and let the difference of the levels of A' and $a' = h'$. Let M' be the space occupied by the air, measured as before. Its elastic force $= (H - h) . \frac{M}{M'}$. To this let h' be added, the difference of the heights of the two columns of mercury; and the pressure of the air in the receiver on the mercury $= h' + (H - h) . \frac{M}{M'}$.

Let $S =$ the space which the condensed air in the receiver would have occupied, if pressed only by the atmosphere, and $S' =$ its actual magnitude; then (458)

$$\frac{S}{S'} = \frac{p}{H}, \quad \text{and if } S = nS',$$

$$n = \frac{h'}{H} + \frac{H - h}{H} . \frac{M}{M'},$$

which will determine the condensation of the air.

SECT. XI.

633. To find the velocity with which air is impelled by the weight of the atmosphere into an exhausted receiver.

Let $h =$ the height of a column of water in equilibrio with the weight of the atmosphere, and s and $s' =$ the specific gravities of water and air respectively. The height of a column of air in equilibrio with the weight of the atmosphere, and of the same density with the air at the earth's surface $= \frac{s}{s'} . h$. And since the elastic force of the air is equal to the force of compression; air thus compressed will move with a velocity equal to that acquired in falling down $\frac{s}{s'} . h$ feet, or to an uniform velocity $= \sqrt{2g . \frac{s}{s'} . h}$ feet in one second.

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This however only expresses the initial velocity: for the air expanding in the space which at first was supposed a vacuum, will oppose a resistance to the issuing air; and at length becoming of the same density with the pressing column, the motion will cease.

634. COR. 1. If a body moves with a greater velocity than this, the space deserted by it must for some time be a vacuum.

635. COR. 2. The velocity with which air issues into an exhausted receiver cannot be increased by any artificial pressure.

For the density of the air will be increased in the same proportion as the pressure; and therefore the velocity which depends on the pressure and the mass moved will be the same.

636. To determine the velocity with which the atmospheric air will rush into a vessel containing rarer air.

Let $s =$ the density of the atmosphere, and $s' =$ that of the rarer air: $P =$ the pressure of the atmosphere; therefore the force impelling the rarer air into a vacuum $= \frac{Ps'}{s}$;

$$\therefore \text{ the moving force} = P - \frac{Ps'}{s}.$$

Let $V =$ the velocity of air rushing into a vacuum,
and $v =$ the velocity with which it will rush into the rarer air;
then since the pressures are as the squares of the velocities of efflux,

$$P : P - \frac{Ps'}{s} :: V^2 : v^2;$$

$$\therefore v = V . \sqrt{\frac{s - s'}{s}}.$$

The resistance is not here considered, which the atmospheric air will meet with from the inertia of that in the vessel, which it must displace in its motion.

Ex. If the density of the rarer air be to that of the atmosphere :: 1 : 4; the velocity with which atmospheric air will rush into the vessel $= \frac{V}{2}\sqrt{3}$.

637. Cor. 1. When the air ceases to flow into the vessel, the density of the air within it is equal to that of the atmosphere.

For if $v = 0$, $s = s'$.

638. Cor. 2. The velocity with which the air rushes into the vessel, continually decreases.

For as the vessel fills, s' increases; $\therefore\ s-s'$ decreases.

639. To determine the time in which the air of the atmosphere will flow into a given vessel before the air in the vessel attains a given density.

Let s and s' represent the densities as before, and $H=$ the height of a homogeneous atmosphere, or the height through which a body must fall to acquire the velocity V. Let $M=$ the capacity of the vessel, and $A=$ the area of the orifice. The quantity of air which would fill the vessel $\propto$ content of the vessel and density of the air jointly, and may therefore be represented by Ms when it is common air, and by Ms' when it is air of the density s'.

Now when t becomes $(t+dt)$, let s' become $(s'+ds')$; therefore the quantity of air admitted in the time dt

$$=M.(s'+ds')-Ms'=Mds'.$$

But when the air in the vessel has the density s', the velocity with which air rushes in $=V\sqrt{\dfrac{s-s'}{s}}=\sqrt{2gH.\dfrac{s-s'}{s}}$;

therefore the quantity admitted in the time dt

$$=V.\sqrt{\frac{s-s'}{s}}\,.\,sA\,dt=\sqrt{2gH}.\sqrt{s-s'}.\sqrt{s}A\,dt;$$

which $=Mds'$;

$$\therefore\ dt=\frac{M}{A\sqrt{2gHs}}\times\frac{ds'}{\sqrt{s-s'}},$$

$$\text{whence }\ t=\frac{-M}{A\sqrt{2gHs}}\times 2\sqrt{s-s'}+C$$

$$=\frac{2M}{A\sqrt{2gHs}}.(\sqrt{s}-\sqrt{s-s'}),$$

since $t=0$ when $s'=0$.

640. Cor. To determine the time in which the vessel will be filled.

Here s and s' must be equal;

$$\therefore\ t = \frac{2M}{A\sqrt{2gHs}} . \sqrt{s} = \frac{2M}{A\sqrt{2gH}}.$$

Ex. To determine the time in which the density of the air in the vessel would be to that of the atmosphere :: 3 : 4.

$$\text{Here } t = \frac{2M}{2A\sqrt{2gH}} = \frac{M}{A\sqrt{2gH}}.$$

641. If the air in a regular vessel be compressed by a weight acting on the cover, which is moveable down the vessel; to determine the velocity with which the air is expelled through an aperture.

Let s be the density of the external air, and P its pressure; and let $p =$ the additional pressure, and s' the density of the air in the vessel when compressed by the action of the weight.

$$\text{Then } P : P + p :: s : s';$$
$$\therefore\ P : p :: s : s' - s,$$
$$\text{or } p = P . \frac{s' - s}{s}.$$

Now the quantities of motion being as the forces which similarly produce them;

$$\therefore\ P : P . \frac{s' - s}{s} :: M'V : m'v,$$

where M' and m' represent the quantities of matter expelled, V the velocity with which air rushes into a vacuum, and v the velocity required. But since the quantities issuing from the same orifice in a small given time are as the densities and velocities jointly;

$$\therefore\ sV^2 : s'v^2 :: M'V : m'v$$
$$:: s : s' - s;$$

$$\text{and } v = V . \sqrt{\frac{s' - s}{s'}}.$$

642. Cor. Since $P : P+p :: s : s'$;

$$\therefore\ p : P+p :: s'-s : s',$$

$$\text{whence } v = V \,.\, \sqrt{\frac{p}{P+p}}.$$

643. In the preceding Propositions the motion of the air has been considered as produced by its weight only. In the following the effects of its elasticity are considered.

644. To determine the velocity of air issuing from a small orifice, supposing it to pass into a vacuum.

Let P = the pressure of the atmosphere; it will be equal to the expelling force at first, since it will counterbalance the initial elasticity. Let s and s' be the density at first, and at the end of t''; V and v the velocity at first, and at the end of t''; M' and m' the quantities issuing in equal times in the two cases.

The elastic force of the air at the end of $t'' = \frac{Ps'}{s}$.

And the moving forces varying as the quantities of motion produced in equal times,

$$P : \frac{Ps'}{s} :: M'V : m'v.$$

$$\text{But } M' : m' :: sV : s'v;$$

$$\therefore\ s : s' :: sV^2 : s'v^2,$$

$$\text{whence } V = v,$$

or the air in the vessel will always issue into a vacuum with the same velocity.

645. Given the initial density (s) of the air issuing into a vacuum; to determine the time in which it will have any other density (s').

Let H = the height due to the constant velocity V.

then $\sqrt{2gH} \,.\, As' \,.\, dt$ = the quantity discharged in the time dt, when the density is s'.

If therefore $M=$ the capacity of the vessel, Ms may represent the quantity of air at first, and Ms' the quantity at the end of t''; therefore the quantity issued $=Ms-Ms'$,

$$\text{and the quantity issuing in the time } dt=-Mds',$$

$$\text{whence } \sqrt{2gH}\,.\,As'\,.\,dt=-Mds';$$

$$\therefore\ dt=-\frac{M}{A\sqrt{2gH}}\cdot\frac{ds'}{s'};$$

$$\therefore\ t=-\frac{M}{A\sqrt{2gH}}\,.\,\text{hyp.log.}\,s'+C=\frac{M}{A\sqrt{2gH}}\times\text{hyp.log.}\,\frac{s}{s'}.$$

646. Cor. If $s'=0$, hyp. log. $\frac{s}{s'}$ becomes infinite; that is, the time would be infinite in which the vessel would empty itself into a vacuum.

647. To determine the velocity with which condensed air would issue from a small orifice in a vessel into a circumambient air of less density but of infinite extent and continuing of the same density.

Let $s=$ the density of the circumambient air, and $F=$ its elastic force; $s'=$ the density of the air in the vessel; and $\therefore\ \frac{Fs'}{s}=$ its initial elastic force; $\sigma=$ the density after t'', and $\therefore\ \frac{F\sigma}{s}=$ its elastic force. And let V, v, M', m', be taken as before (644).

$$\text{The initial expelling force}=\frac{Fs'}{s}-F=F\,.\,\frac{s'-s}{s},$$

$$\text{and the force after } t'' \text{ is}=F\,.\,\frac{\sigma-s}{s}.$$

But these forces are as the quantities of motion generated in equal times;

$$\therefore\ F.\frac{s'-s}{s} : F.\frac{\sigma-s}{s} :: M'V : m'v :: s'V^2 : \sigma v^2;$$

$$\therefore\ v=V.\sqrt{\frac{s'.(\sigma-s)}{\sigma.(s'-s)}}.$$

648. Cor. 1. If $\sigma=s$, $v=0$, or the air ceases to issue from the vessel.

649. Cor. 2. If $s'=s$, the initial expelling force $=0$; $\therefore$ $V=0$, and the air will not issue from the vessel.

650. Given the density (s') of the air condensed in a vessel; to determine the time in which its density will become σ, when it issues through a small orifice into a medium of less density but of infinite extent.

Let $H=$ the height due to the initial velocity V.

Since the heights due to the velocities are as the squares of those velocities, the height due to the velocity v is

$$=H.\frac{s'.(\sigma-s)}{\sigma.(s'-s)}.$$

Hence the quantity of air issuing in the time dt

$$=\sqrt{2gH}.A\sigma dt.\sqrt{\frac{s'.(\sigma-s)}{\sigma.(s'-s)}};$$

$$\text{But it also } =-Md\sigma,$$

$$\text{whence } dt=\frac{M.\sqrt{s'-s}}{A\sqrt{2gHs'}}.\frac{-d\sigma}{\sqrt{\sigma^2-s\sigma}};$$

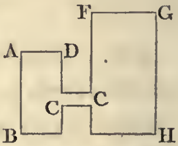
$$\therefore\ t=\frac{M\sqrt{s'-s}}{A\sqrt{2gHs'}}\times\text{hyp. log.}\left(\frac{1}{\sigma-\frac{1}{2}s+\sqrt{\sigma^2-s\sigma}}\right)+C$$

$$=\frac{M\sqrt{s'-s}}{A\sqrt{2gHs'}}\times\text{hyp. log.}\left(\frac{s'-\frac{1}{2}s+\sqrt{s'^2-ss'}}{\sigma-\frac{1}{2}s+\sqrt{\sigma^2-s\sigma}}\right).$$

651. Cor. If $\sigma=s$, the air ceases to issue;

$$\therefore\ \text{the whole time}=\frac{M\sqrt{s'-s}}{A\sqrt{2gHs'}}\times\text{hyp. log.}\frac{s'-\frac{1}{2}s+\sqrt{s'^2-ss'}}{\frac{1}{2}s}.$$

652. If two cylinders contain airs of different densities, and communicate by a small orifice, to determine the velocity with which the air will flow from one to the other.



Let $ABCD$ contain the denser air, which therefore will flow into $FCHG$. Let $s =$ the density of the atmosphere, and $F =$ its elastic force;

$s' =$ the density of the air in $ABCD$ at first, and $\therefore F.\frac{s'}{s} =$ its elastic force.

$s'' =$ the density after t'', and $\therefore F.\frac{s''}{s} =$ its elastic force.

$\sigma =$ the density of the air in $FCHG$ at first, and $\therefore F.\frac{\sigma}{s} =$ its elastic force.

$\sigma' =$ the density after t'', and $\therefore F.\frac{\sigma'}{s} =$ the elastic force.

$V =$ the velocity at first, and $v =$ velocity after t''.

At first the expelling force of the air in $ABCD = F.\left(\frac{s'}{s} - \frac{\sigma}{s}\right)$,

$$\text{and after } t'', \text{ it} = F.\left(\frac{s''}{s} - \frac{\sigma'}{s}\right);$$

$$\therefore \ F.\frac{s' - \sigma}{s} : F.\frac{s'' - \sigma'}{s} :: s'V^2 : s''v^2;$$

$$\therefore v = V.\sqrt{\frac{s'.(s'' - \sigma')}{s''.(s' - \sigma)}}.$$

653. Cor. 1. If $s'' = \sigma'$, the air will cease to flow from one vessel into the other.

654. Cor. 2. The whole quantity of air in the two vessels continuing the same, let $A =$ the capacity of $ABCD$, and B of $FCHG$; then

$$As' + B\sigma = As'' + B\sigma';$$

$$\therefore \sigma' = \frac{A.(s' - s'') + B\sigma}{B},$$

$$\text{whence } v = V.\sqrt{\frac{s'.\{B.(s'' - \sigma) - A.(s' - s'')\}}{Bs''.(s' - \sigma)}}.$$

655. Cor. 3. To determine the time in which the density becomes s''.

$$\text{Since } v = V.\sqrt{\frac{(As' + Bs').s'' - (As' + B\sigma).s'}{B.(s' - \sigma).s''}}.$$

$$\text{Let } As' + Bs' = m, \quad As' + B\sigma = n, \quad B(s' - \sigma) = r;$$

$$\therefore \sqrt{2gH}.Os''.dt\sqrt{\frac{ms'' - ns'}{rs''}} = -Mds'',$$

$$\text{and } dt = -\frac{M\sqrt{r}}{O\sqrt{2gH}}.\frac{ds''}{\sqrt{ms''^2 - ns's''}},$$

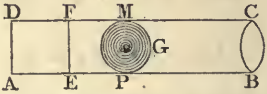
whence

$$t = \frac{M\sqrt{r}}{O\sqrt{2gHm}} \times \text{hyp. log.} \left(\frac{s' - \frac{ns'}{2m} + \sqrt{s'^2 - \frac{ns'^2}{m}}}{s'' - \frac{ns'}{2m} + \sqrt{s''^2 - \frac{ns's''}{m}}} \right).$$

656. In the preceding Propositions, the air has been supposed to continue of the same temperature, and its elasticity dependent upon the density only: whereas heat is found to have a considerable influence on its elastic force. When a given quantity has acquired an increase of heat, it will acquire a greater elastic force, and therefore expand or exert a greater pressure upon whatever may prevent its expansion. This increase of elastic force, however, being of the same nature with that which would be caused by a greater condensation of air which continues of the same temperature, the effects produced in the two cases may be compared. Hence the action of any elastic fluid may be compared with that of condensed

air; as for instance, the elastic force of condensed air in the *Air-Gun*, with that of the fluid produced by the inflammation of gunpowder; though the intensity of the one vastly exceeds that of the other.

657. If a sphere be placed in a horizontal cylinder of the same diameter, without friction, and be driven by the action of condensed air, or the elastic fluid generated by the inflammation of gunpowder; to determine the velocity of the sphere in any position.



Let PM be the position of the ball, and $AEFD$ the space occupied by the air at first; and let $s' =$ its density, s being $=$ that of the atmosphere, and $F =$ the elastic force; therefore the elastic force of the air in $AEFD = F \, . \, \frac{s'}{s}$.

Let $r =$ the radius of the sphere or cylinder, $AE = b$, $AP = x$, and the velocity at $P = v$.

When the sphere has moved from E to P, the air which occupied $AEFD$, now occupies $APMD$, and therefore its elastic force is $F \, . \, \frac{s'}{s} \times \frac{\pi a^2 b}{\pi a^2 x} = \frac{F s' b}{s x}$.

Hence the force impelling the sphere towards B is $= F \, . \, \frac{s' b}{s x} - F$;

$$\therefore \; G v d v = F \, . \left(\frac{s' b}{s x} - 1 \right) . \, d x,$$

$$\text{and } v^2 = \frac{2F}{G} \, . \left\{ \frac{s' b}{s} \, . \, \text{hyp. log.} \, \frac{x}{b} + (b - x) \right\}.$$

658. Cor. 1. If $x = AB = l$,

$$V^2 = \frac{2F}{G} \, . \left\{ \frac{s' b}{s} \, . \, \text{hyp. log.} \, \frac{l}{b} + (b - l) \right\}.$$

659. Cor. 2. If $\frac{s'}{s}$ be known, or the ratio of the densities of the air in *AEFD* and atmospheric air, or the ratio of the initial elastic force of the fluid and atmospheric air, V may be determined, or the converse.

660. Cor. 3. To determine the length l, which will give the velocity of a ball issuing from the mouth of a cannon, a maximum.

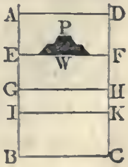
$$\frac{s'b}{s}.\text{hyp. log.}\frac{l}{b}+(b-l)=\text{max.}$$

$$\therefore \frac{s'b}{s}\times\frac{dl}{l}-dl=0,$$

$$\text{and} \quad l=\frac{s'b}{s}.$$

This is a little greater than what would result from experiment. But it may be observed that the elastic force of the fluid has been supposed to be inversely as the space occupied, which may not be quite correct. Neither is the fluid arising from the inflammation of the gunpowder homogeneous; nor is it all inflamed at once; and some will escape between the ball and sides of the cannon. The longer also the cannon is, the longer will the resistance continue which is caused by friction.

661. If in a closed vertical cylinder, the upper part of which is exhausted, a quantity of air be compressed by a weight acting upon its lid, which is moveable, without friction, and air-tight; and an additional weight be placed upon the former; to determine the velocity with which it will descend.



Let *ABCD* be the cylinder, the upper part *AEFD* of which is exhausted, and the air kept in *EBCF* by a weight W acting upon the lid *EF*. Let P be the additional weight, and *GH* the position of the lid at the end of t''. Let $EB=a$, $EG=x$, and $v=$ the velocity of $P+W$.

Then since the air in $EBCF$ is kept there by the weight W, its elastic force $= gW$;

$$\therefore \text{ the elastic force of } GBCH = gW \cdot \frac{a}{a-x},$$

and the absolute force causing $P + W$ to descend

$$= g \cdot (P + W) - gW \cdot \frac{a}{a-x}.$$

Hence $vdv = g \cdot \left\{1 - \frac{W}{P+W} \cdot \frac{a}{a-x}\right\} \cdot dx$,

$$\text{and } v^2 = 2g \cdot \left\{x + \frac{Wa}{P+W} \cdot \text{hyp. log. } \left(\frac{a-x}{a}\right)\right\}.$$

662. Cor. 1. Since whilst x increases from 0, the first term of the value of v increases whilst the second decreases;

$$\therefore v \text{ will become} = 0, \text{ when } x + \frac{Wa}{P+W} \cdot \text{hyp. log.} \frac{a-x}{a} = 0,$$

$$\text{or } x \cdot (P + W) = aW \cdot \text{hyp. log.} \frac{a}{a-x}.$$

When therefore the lid has descended to a certain distance, determined from this equation, which suppose $= EI$, it will ascend again to its first position, and then descend; and so on.

663. If the cylinder be open at the top and communicate with the atmosphere, and the lower part contain air in its natural state, the elastic force of which is equal to the pressure of the atmosphere upon the lid; and a weight W be placed on the lid; to determine its velocity at any point, the lid descending parallel.

Let $EB = a$, $EG = x$, and v the velocity of the descending body.

Let gW' represent the pressure of the atmosphere on the lid, or the elastic force of the air in $EBCF$. When the lid is at GH, the elastic force of the air $GBCH$ is $= gW' \cdot \frac{a}{a-x}$, and the force impelling the lid is $gW' + gW - gW' \cdot \frac{a}{a-x}$,

and this communicating motion to W,

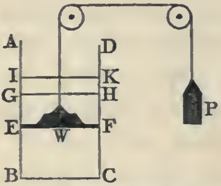
$$vdv = g \cdot \left\{\frac{W + W'}{W} - \frac{W'}{W} \cdot \frac{a}{a-x}\right\} \cdot dx;$$

$$\therefore\ v^2 = 2g \cdot \left\{\frac{W + W'}{W} \cdot x + \frac{W'a}{W} \cdot \text{hyp. log.}\, \frac{a-x}{a}\right\}.$$

664. Cor. 1. Proceeding as in the Cor. to the preceding Prop. it appears that the lid will descend and ascend alternately, as before.

665. Cor. 2. The elasticity of the air being diminished by cold, and increased by heat, it is evident if by any mechanical contrivance the air in *EBCF* can be cooled and then again heated, and so on alternately, that the lid will descend and ascend alternately without the aid of the weight *W*. A machine therefore may be constructed, whose motion may be produced and continued by the action of air or an elastic fluid, cooled and heated alternately. And such is the principle of the *Steam-Engine.*

666. If the vertical cylinder be always open at the top, and contain air in its natural state in the lower part, and the lid be raised by means of a weight attached to a string supposed without weight; to determine the velocity of the weight in any position of the lid.



Let $EB = a$, $EG = x$, and $v =$ the velocity of the descending weight P. Let the pressure of the atmosphere on EF, or the elastic force of the air $EBCF$ be represented by gW. Then when the air occupies $GBCH$, its elastic force is

$$= gW \cdot \frac{a}{a+x};$$

$$\therefore \text{ the moving force} = gP + gW \cdot \frac{a}{a+x} - gW,$$

$$\text{and } v\,dv = g\,.\left\{1 + \frac{W}{P}\,.\,\frac{a}{a+x} - \frac{W}{P}\right\}.\,dx;$$

$$\therefore\ v^2 = 2g\,.\left\{\frac{a\,W}{P}\,.\text{ hyp. log. }\left(\frac{a+x}{a}\right) - \frac{W-P}{P}\,.\,x\right\}.$$

667. Cor. 1. If W be greater than P, as x increases, both terms in the value of v increase, and therefore v decreases. After ascending therefore to a certain height, as EI, the lid will descend, and so on alternately.

668. Cor. 2. If instead of using the weight P, the air in $EBCF$ was to become heated and then cooled, and so on alternately; the alternate motions of ascent and descent would be caused.

SECT. XII.

669. DEF. A TUBE of glass whose diameter is in-considerable is called a *Capillary Tube.*

The internal diameter may vary from $\frac{1}{20}$th to $\frac{1}{50}$th of an inch.

670. There is an attraction of cohesion between glass and water.

For if a smooth plate of glass be brought into contact with water, and withdrawn gently from its surface, a portion of the fluid will adhere to the glass, and remain suspended on its lower surface when placed in a horizontal position. An attraction therefore must exist, and be such at least as is sufficient to sustain the gravity of the water.

Also, if a plate of glass, suspended horizontally from one of the scales of a balance, be kept in equilibrio by a weight in the opposite scale, and then brought into contact with the surface of water, the force of cohesion between them is found sufficient to sustain an additional weight in the other scale. This cohesion is not produced by the air's pressure; for the same effect takes place *in vacuo.* Hence an attraction must exist between the particles of the fluid and solid.

671. The constituent particles of water have an attraction towards each other.

For in the preceding experiment, when the plate is withdrawn, a thin stratum of fluid adheres to it. And it is found that the force employed to detach this from the rest of the fluid is far greater than its weight. An attraction therefore necessarily exists which would keep the thin stratum united to the rest of the fluid, independently of its weight, or the particles have an attraction towards each other.

672. The particles of mercury have a strong attraction towards each other.

This appears from the spherical form which a small portion of that fluid assumes, and from the resistance which it opposes to any separation of its parts.

Also if a quantity of mercury be separated into a number of small parts, all these will be spherical; and if any two of them be brought into contact, they will instantly rush together, and make a single drop of the same form.

673. The attraction between glass and water is imperceptible unless the distance between them be very small.

For whatever be the thickness of the plate, *cæt. par.* the force employed to detach it from the water is the same. Hence any new laminæ of matter that may be added to the plate exert no action on the fluid; whence it is inferred, that the indefinitely thin lamina of water attached to the surface of the plate interposes between the plate and the rest of the fluid a distance sufficiently great to prevent any sensible effect from their mutual attraction, and that the force necessary to detach all laminæ of fluid of the same dimensions is equal; being that which is sufficient to separate the fluid from itself.

Also water, if its temperature be the same, is found to rise to the same height in capillary glass tubes of the same internal diameter, whatever be their thickness. The laminæ therefore of the glass tubes, however small their distance from the interior surface, do not contribute to the ascent of the water.

When the interior surface of a capillary tube is lined with a *very thin* coating of an unctuous substance, the water will no longer ascend. Now if the attraction of a glass tube were similar to the attraction of gravity, it would act through media of all kinds, and in this case cause the fluid in which it is immersed, to rise. But as the intervention of a thin lamina of grease destroys capillary action, we have reason to infer that it does not extend to sensible distances.

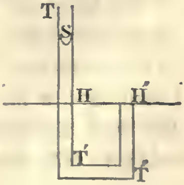
674. From the preceding and similar facts, it is inferred that the attraction of a capillary tube, when it exceeds the attraction of the fluid on itself, exerts its influence only in raising the water immediately in contact with it; which being thus raised, and forming an interior tube through its own attraction on the water, raises and supports what is next in contact with itself.

675. The inclination or direction of the first elements of the fluid, that is, those in contact with the solid, where their mutual affinity is sensibly exerted, depends upon the respective natures of the fluid and solid. When these are the same, it is found to be invariable whatever be the form of the attracting surface of the solid, whether, for instance, it be a tube or a plane. But beyond these first elements, and out of the sphere of sensible activity of the solid body, the direction of the other elements and the form of the surface are solely determined by the action of the fluid on itself.

676. From experiments and micrometrical measurements it has been inferred that when the water moves freely in a capillary tube, the surface will be a hemisphere nearly touching the interior surface of the tube. And when it rises between two planes, it has a circular form, and is a tangent to the planes.

677. To determine the force by which water is raised in the tube.

Let *HT* be a vertical cylindrical tube of glass, having its base, which is horizontal, immersed in water. Imagine the tube produced to T', then carried horizontally to T'', and vertically to H'', and let the fluid rise to S. This imaginary



tube $HT''T''H''$ being supposed to have no thickness, will neither attract nor repel the fluid particles, and therefore not alter the circumstances of the problem. The fluid in the tube $ST'T''H''$ being in equilibrio, the column ST' balances the column $H''T''$, and therefore the sum of the vertical forces acting on the column ST' is equal to the sum of those acting on $H''T''$. In estimating them we may neglect the attractions of the fluid *in* either column ST' or $H''T''$ on itself, because the attractions are equal and opposite.

The forces acting on the fluid in HT' are gravity, the attraction of the exterior fluid surrounding the imaginary tube, and the attraction of the glass in the real tube. The two former of these forces are balanced by equal forces acting on the fluid in $H''T''$; there remains therefore the attraction of the glass which raises the fluid in HT'. Call this force F.

The forces acting on the fluid in HS are gravity, the attraction of the exterior fluid surrounding the imaginary tube HT', and the attraction of the glass. The last arises from that part only of the tube which is above the column HS; for if the column HS be supposed divided into an indefinite number of vertical columns, and through the upper extremity of each of these a horizontal plane be drawn, it is evident that the part of the tube between this plane and the surface HH'' will produce equal and opposite vertical forces on the column. The vertical force therefore on the column HS arises from the glass above S, and is the same with the force of the whole real tube on the column HT' which is of the same diameter as HS, and similarly situated in the imaginary tube. This force therefore will be F. Let F' = the attraction on HS arising from the exterior fluid surrounding the imaginary tube; then $2F - F'$ − the gravity of HS will represent the sum of the vertical forces on the column HS, independent of those which are balanced by the forces on $H''T''$. Hence $2F - F' = g\,DM$; if g represent the force of gravity, M the magnitude of the column, and D the density.

678. Cor. 1. M will always have the same sign with $2F - F'$; and as this quantity is positive, nothing, or negative, the interior level will be higher, equal with, or lower than the exterior.

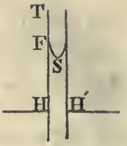
679. Cor. 2. If the attractions of matter of different kinds be expressed by the same function of the distance, without any other modification than a constant coefficient ρ, ρ' which may serve to measure their intensity; the attractions of two particles equally distant from the same attracted particle would be proportional to ρ, ρ'; and this proportionality would still hold for masses of similar figures attracting a particle similarly placed. In this case $+F$ and $-F'$ would be proportional to the intensities of the attractions exerted by the tube on the fluid and the fluid on itself. Now the attracting forces being sensible only at imperceptible distances, the material tube HT will act sensibly only on the columns extremely near. The consideration therefore of the curvature may be neglected, and the inner surface be supposed to be developed on a plane. F therefore will be proportional to the width of this plane or inner circumference of the tube (C);

$$\therefore\ F = \rho C,$$

ρ representing the intensity of attraction of the matter of the tube. In the same manner $F' = \rho' C$, ρ' representing the intensity of attraction of the fluid on itself. Hence

$$g\,DM = (2\rho - \rho') \,.\, C.$$

680. To determine the elevation of water in cylindrical capillary tubes, the column being terminated by a hemisphere.



Let r = the radius of the innner surface,

h = the height of the column HS;

$\therefore$ $2\pi r$ = the circumference of the inner surface,

and $\pi r^2 h$ = the magnitude of the cylinder from H to S.

In order to have the whole mass, we must add the meniscus which terminates the column, and which is equal to the difference between a cylinder of the same base and altitude r, and a hemisphere whose radius is r; that is, it

$$= \pi r^3 - \tfrac{2}{3}\pi r^3 = \tfrac{1}{3}\pi r^3;$$

$$\text{whence } M = \pi r^2 h + \tfrac{1}{3}\pi r^3,$$

$$\text{and } gD \,.\, \pi r^2 \,.\, (h + \tfrac{1}{3} r) = (2\rho - \rho') \,.\, 2\pi r;$$

$$\therefore\ h + \tfrac{1}{3} r = 2 \,.\, \frac{(2\rho - \rho')}{gD} \,.\, \frac{1}{r},$$

which is applicable to all cylindrical tubes where the fluid is terminated by a hemisphere.

681. Cor. 1. If we compare any tubes of the same nature, plunged into the same fluid, at a constant temperature; ρ, ρ', g, D will be the same for each, and $\therefore$ $2 \,.\, \dfrac{2\rho - \rho'}{gD}$ is invariable: let this $= A$;

$$\therefore\ h + \tfrac{1}{3} r = \frac{A}{r}.$$

682. Cor. 2. If a = the mean altitude, or that at which the fluid would stand if formed into a cylinder,

$$M = \pi r^2 a;$$

$$\therefore\ \pi r^2 a = \pi r^2 h + \tfrac{1}{3}\pi r^3,$$

$$\text{or } a = h + \tfrac{1}{3} r;$$

that is, the mean height is equal to the height of the lowest point of the concavity of the meniscus, increased by $\frac{1}{3}$ of the radius of the tube.

683. Cor. 3. Hence the mean height varies inversely as the radius of the tube, which agrees nearly with observation.

684. To determine the elevation of a fluid between two plane laminæ of considerable extent, whose distance $= b$.

When the planes are of a nature to be wetted by the fluid, the fluid prism may be considered as terminated by a horizontal cylinder whose diameter $= b$, and whose length is equal to that of the planes, which call a. Here indeed the curvature of the fluid at the two extremities of the cylinder is neglected; which may be done, if the planes be supposed to be indefinitely extended, or the formula applied only to portions of the surface sufficiently distant from the extremities of the plane to be sensibly cylindrical.

From H to S is a rectangular parallelopiped whose horizontal circumference $C = 2a + 2b$, the surface of its base $= ab$, and its magnitude $= abh$.

The meniscus is equal to the difference between a small rectangular prism (whose base is $= ab$, and altitude $= \frac{1}{2} b$) and a circular semi-cylinder whose length is $= a$ and radius $= \frac{1}{2} b$; that is, it is $= \frac{1}{2} a b^2 - \frac{1}{8} . \pi a b^2 = \frac{1}{2} a b^2 . (1 - \frac{1}{4} \pi)$;

$$\therefore\ gD . \{abh + \tfrac{1}{2} a b^2 . (1 - \tfrac{1}{4}\pi)\} = 2 . (2\rho - \rho') . (a + b),$$

$$\text{or}\ \ b . \{h + \tfrac{1}{2} b . (1 - \tfrac{1}{4}\pi)\} = \frac{2 . (2\rho - \rho')}{gD} . \left(1 + \frac{b}{a}\right).$$

685. Cor. 1. If the extent of the planes be indefinitely great, $\frac{b}{a}$ may be considered as evanescent;

$$\therefore\ b\,.\left\{h+\tfrac{1}{2}b\,.(1-\tfrac{1}{4}\pi)\right\}=\frac{2\,.(2\rho-\rho')}{gD}=A,$$

$$\text{and }\ h+\tfrac{1}{2}b\,.(1-\tfrac{1}{4}\pi)=\frac{A}{b}.$$

686. Cor. 2. If a' be the mean altitude, or that at which the fluid would stand if formed into a parallelopiped,

$$M=aba',$$

whence as before, $a'=h+\frac{1}{2}b\,.(1-\frac{1}{4}\pi)$.

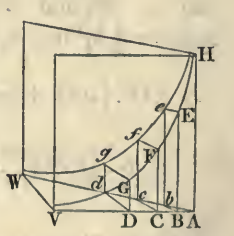
687. Cor. 3. Hence the mean altitude varies inversely as the distances between the planes.

688. Cor. 4. Since the mean altitude is nearly $=\frac{A}{b}$; and in capillary tubes it is $=\frac{A}{r}$, where A is the same for the planes and the tubes, if they are of the same nature, it is evident that the mean height of the fluid between two planes whose distance $=b$, is the same as in a cylindrical tube whose radius $=b$.

689. If the angle, at which two planes meet, be very small; the figure of the water in contact with either plane will be a common hyperbola.

Let the planes *HAV*, *HAW*, inclined at a very small angle, be immersed so that their common section *HA* may be perpendicular to the surface of the fluid *WAV*.

Let *HEFGV*, *HefgW* represent the figures of the water in contact with either plane. Take any number of points *B*, *C*, *D* in the line *AV*; and in the plane *HAV* draw *BE*, *CF*, *DG* perpendicular to *AV*, and therefore parallel to *HA*, and perpen-



dicular also to the plane VAW (Euc. XI. 8). Also from the same points and in the plane VAW draw Bb, Cc, Dd perpendicular to AV, and therefore parallel to each other. And from b, c, d, in the plane HAW draw be, cf, dg perpendicular to AW, and therefore parallel to AH and perpendicular to the plane VAW; hence they will be parallel to EB, CF, DG. Since therefore the angle VAW is small, the fluid in each section may be considered as raised by parallel planes, and therefore the altitude varies inversely as the distance (687), or

$$BE : CF :: Cc : Bb :: AC : AB,$$

since Bb is parallel to Cc;

and this is the property of the common hyperbola whose asymptotes are AV, AH.

◆

ERRATA.

Page 44 line 11, *for* $\{a^2-a^2x^2\}^{\frac{3}{2}}$ *read* $\{a^2-c^2x^2\}^{\frac{3}{2}}$.

73 3 from the bottom, *for* MD *read* MS.

80 2 from the bottom, *for* bodies *read* body.

87 11, for $n\sqrt{n-1}$, read $\sqrt{n-1}$.

105 20,
— 21, } *for* $\sin n$ *read* $\sin(m+n)$.

128 last line,
129 line 1,
— 2, } for $\frac{Ag\sqrt{t}}{k}$ read $\frac{t\sqrt{Ag}}{k}$.

134 19, for $edhg$ read $cdhg$.

139 5 from the bottom *for* EGF *read* $2GF$.

151 13, *for* $+\frac{1}{2}\cdot\frac{O^2g}{S^2}\cdot t^2$ *read* $-\frac{1}{2}\cdot\frac{O^2g}{S^2}\cdot t^2$.

156 1, *for* $\frac{4}{3}\cdot(2r)^{\frac{3}{2}}$ *read* $\frac{4}{3}r\cdot(2r)^{\frac{3}{2}}$.

172 19, for $-Na-M$ read $-Na-Mb$.

185 15, *dele* t.

230 18, *add* $=0$.

244 2 from the bottom, *read* $\int\frac{-dz}{\sqrt{\cos z-\cos z'}}$.

247 21, *add* inversely.

249 15, after cube, *add* jointly.

306 19, *for* $\frac{1}{d^n}$ *read* $\frac{1}{d^4}$.

351 1. *for* considerable *read* inconsiderable.